Show all work and justifications for each problem in a logical format.

1. Is any number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

$n = n^3+2$
$0 = n^3-n+2$
$f(n) = n^3-n+2$
$f(-2) = -4$
$f(0) = 2$

$f(n)$ is a polynomial function → Continuous

Since $f(n)$ is continuous on $[-2,0]$ and $f(-2)<0<f(0)$, by the IVT there exists $x=c$ on $(-2,0)$ such that $f(c)=0$. ∴ $0=c^3-c+2$ and $c=c^3+2$.

$c \approx -1.521$

2. Show that for all spheres with radii in the interval $[1,5]$ there is one with a volume of 275 cubic centimeters.

$V = \frac{4}{3}\pi r^3$
$V(1) = \frac{4\pi}{3} \approx 4.19$
$V(5) = \frac{4\pi}{3}(5)^3 = \frac{500\pi}{3} \approx 523.6$

V is a polynomial function → Continuous

Since V is continuous on $[1,5]$ and $V(1)<275<V(5)$, by the IVT, there exists $x=c$ on $(1,5)$ such that $f(c) = 275\ cm^3$.

$r \approx 4.0341\ cm$

3. Prove that the function $f(x) = \cos x - \sin x^2$ has a zero on the interval $[0,\pi]$.

$f(0) = 1$
$f(\pi) = -1$

$f(x)$ is continuous since it is the composition and difference of continuous functions (I.e., $\cos(x)$, $\sin(x)$, x^2)

Since $f(x)$ is continuous on $[0,\pi]$ and $f(\pi)<0<f(0)$, by the IVT, there must exist $x=c$ on $(0,\pi)$ such that $f(c)=0$. ∴ $\cos c - \sin c^2 = 0$ or $f(x)$ has a zero at $x=c$.

4. Prove that there is a positive number c such that $c^2 = 2$. This proves the existence of the number $\sqrt{2}$.

If $x^2=2$
$x^2-2=0$
$f(x)=x^2-2$
$f(0)=-2$
$f(2)=2$

$f(x)$ is a polynomial function → Continuous

Since $f(x)$ is continuous and $f(0)<0<f(-2)$, by the IVT, there must exist $x=c$ on $(-2,2)$ such that $f(c)=0$. ∴ $c^2-2=0$ and $c^2=2$