Please show all work on a <u>separate sheet of paper</u>.

Be sure to complete one of each type of problem in class to ensure understanding!

1. Find the derivatives of the following functions:

a. $f(x)=\dfrac{5x^3+6x-10}{\sqrt[3]{x}}$

$f(x)=5x^{\frac{8}{3}}+6x^{2/3}-10x^{-1/3}$

$f'(x)=\frac{40}{3}x^{5/3}+4x^{-1/3}+\frac{10}{3}x^{-4/3}$

$\boxed{f'(x)=\frac{40}{3}x^{5/3}+\frac{4}{x^{1/3}}+\frac{10}{3x^{4/3}}}$

b. $f(x)=\dfrac{5x^3+6x-10}{x^2+2}$

$f'(x)=\dfrac{(15x^2+6)(x^2+2)-2x(5x^3+6x-10)}{(x^2+2)^2}$

$f'(x)=\dfrac{15x^4+36x^2+12-10x^4-12x^2+20x}{(x^2+2)^2}$

$\boxed{f'(x)=\dfrac{5x^4+24x^2+20x+12}{(x^2+2)^2}}$

c. $f(x)=\dfrac{x^2-10x+25}{x-5}$

$f(x)=x-5$

$\boxed{f'(x)=1}$

d. $f(x)=-2(3x+7)^2$

$f'(x)=-2(2)(3x+7)\cdot 3$

$f'(x)=-12(3x+7)$

$\boxed{f'(x)=-36x-84}$

e. $f(x)=\dfrac{\cos x}{x^2+2}$

$f'(x)=\dfrac{-\sin x(x^2+2)-2x\cos(x)}{(x^2+2)^2}$

$\boxed{f'(x)=\dfrac{-x^2\sin x-2\sin x-2x\cos x}{(x^2+2)^2}}$

f. $f(x)=\sin^2(5x)-\cos^2(5x)$

$=-1(\cos^2(5x)-\sin^2(5x))$

$=-\cos(10x)$

$f'(x)=-(-\sin(10x))\cdot 10$

$\boxed{f'(x)=10\sin(10x)}$

g. $f(x)=\sqrt[5]{\cos^2(\pi x)}$

$=(\cos(\pi x))^{\frac{2}{5}}$

$f'(x)=\frac{2}{5}(\cos(\pi x))^{-3/5}(-\sin(\pi x))\pi$

$\boxed{f'(x)=\dfrac{-2\pi\sin(\pi x)}{5(\cos(\pi x))^{3/5}}}$

h. $f(x)=\sin^2(5x)+\cos^2(5x)$

$=1$

$\boxed{f'(x)=0}$

i. $f(x)=\dfrac{1}{(-3(x+2))^3}$

$f(x)=\frac{-1}{27}(x+2)^{-3}$

$f'(x)=-\frac{1}{27}(-3)(x+2)^{-4}$

$\boxed{f'(x)=\dfrac{1}{9(x+2)^4}}$

$12x(4x^3+x$

j. $h(x)=(4x^3+x)^2(12x^2+1)$

$h'(x)=2(4x^3+x)(12x^2+1)(12x^2+1)+(4x^3+x)^2(24x)$

$h'(x)=2(4x^3+x)(144x^4+24x^2+1+48x^4+12x^2)$

$\boxed{h'(x)=2(4x^3+x)(192x^4+36x^2+1)}$

k. $h(x)=9\csc x+3\cot x$

$\boxed{h'(x)=9\csc x\cot x-3\csc^2 x}$

l. $g(x)=\dfrac{1+\tan x}{1-\tan x}$

$g'(x)=\dfrac{\sec^2x(1-\tan x)+\sec^2x(1+\tan x)}{(1-\tan x)^2}$

$=\dfrac{\sec^2x(2)}{(1-\tan x)^2}$

$\boxed{g'(x)=\dfrac{2\sec^2x}{(1-\tan x)^2}}$

2. Which of the following represents $\dfrac{d}{dx}(h(x))$ given $h(x)=\dfrac{\sec(2x)}{\tan(2x)}$?

$h'(x)=\dfrac{2\sec(2x)\tan^2(2x)-2\sec^2(2x)\sec(2x)}{\tan^2(2x)}=\dfrac{2\sec(2x)(\tan^2(2x)-\sec^2(2x))}{\tan^2(2x)}$

$=\dfrac{2\sec(2x)(-1)}{\tan^2(2x)}=\dfrac{-2\sec(2x)}{\tan^2(2x)}$

$\boxed{\text{a.}}$ $\dfrac{-2\sec(2x)}{\tan^2(2x)}=$ $\boxed{\text{b.}}$ $-2\cot(2x)\csc(2x)$ $\boxed{\text{c.}}$ $\dfrac{-2\cos(2x)}{\sin^2(2x)}$ $\boxed{\text{d.}}$ $-2\cos(2x)\csc^2(2x)$

$-2\sec(2x)(\cot^2(2x))$

$-2\left(\dfrac{1}{\cos(2x)}\right)\left(\dfrac{\cos^2(2x)}{\sin^2(2x)}\right)$

$\dfrac{-2\cos(2x)}{\sin(2x)}\cdot\dfrac{1}{\sin(2x)}$

$-2\,\dfrac{\cos(2x)}{\sin(2x)}\cdot\dfrac{1}{\sin(2x)}$

$\dfrac{-2\cos(2x)}{\sin^2(2x)}$

$-2\cos(2x)\csc^2(2x)$

3. Discuss the differentiability of the following functions. If a function is not differentiable at a certain point, explain why. Do not use a calculator!

a. $g(x)=\begin{cases}\tan^{-1}x, & x\neq 0\\ 1, & x=0\end{cases}$

$\lim_{x\to 0}\tan^{-1}(x)=\tan^{-1}(0)=0$

$f(0)=1\neq\lim_{x\to 0}g(x)$

$\boxed{\text{non-differentiable } x=0 \text{; discontinuity at } x=0 \Rightarrow \text{nondifferentiable}}$

b. $f(x)=x+\sqrt{x^2}+2$

$=x+|x|+2$

$\boxed{\text{non-differentiable } x=0 \text{; Sharp corner (turn)}}$

$\lim_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\dfrac{x-x+2-2}{x}$

$=\lim_{x\to 0^-}\dfrac{0}{x}=0$

$\lim_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\dfrac{x+x+2-2}{x}$

$=\lim_{x\to 0^+}\dfrac{2x}{x}=2$

$\lim_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}\neq\lim_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}$

c. $h(x)=\sqrt[3]{|x|}$

$\lim_{x\to 0^-}\dfrac{\sqrt[3]{|x|}-0}{x}=\lim_{x\to 0^-}\dfrac{\sqrt[3]{-x}}{x}$

$=\lim_{x\to 0^-}\dfrac{-1}{x^{2/3}}=-\infty$

$\lim_{x\to 0^+}\dfrac{\sqrt[3]{|x|}-0}{x}=\lim_{x\to 0^+}\dfrac{\sqrt[3]{x}}{x}$

$=\lim_{x\to 0^+}\dfrac{1}{x^{2/3}}=\infty$

$\boxed{\text{non-differentiable at } x=0 \text{; cusp}}$

$\cos(-x)=\cos(x)$!

d. $j(x)=3-\sqrt[3]{x}$

$j'(x)=-\frac{1}{3}x^{-2/3}=\frac{-1}{3x^{2/3}}$

$j'(0)=$ undefined

$\boxed{\text{non-differentiable at } x=0 \quad \text{Vertical tangent line}}$

e. $m(x)=3\cos(|x|)=3\cos(x)$

$\boxed{\text{differentiable on } (-\infty,\infty)}$

f. $n(x)=\begin{cases}(x+1)^2, & x\le 0\\ 2x+1, & 0<x<3\\ (4-x)^2, & x\ge 3\end{cases}$

* $\lim_{x\to 0} n(x)=1=f(0)$ ✓

$\lim_{x\to 3^-} n(x)\ne\lim_{x\to 3^+} n(x)$

$\boxed{\text{discontinuity at } x=3 \Rightarrow \text{non-differentiable } x=3}$

* $\frac{d}{dx}\Big|_{x=0}(x+1)^2=2(x+1)\Big|_{x=0}=2=\frac{d}{dx}\Big|_{x=0}(2x+1)=2\Big|_{x=0}=2$ ∴ differentiable at $x=0$

4. Find an equation of the tangent line and normal at the point indicated: $y=\tan^3 x$, $x=\frac{\pi}{4}$.

$y'=3\tan^2 x\sec^2 x$

$f(\frac{\pi}{4})=\tan^3(\frac{\pi}{4})=(1)^3=1$

$y'\big|_{x=\frac{\pi}{4}}=3(\tan(\frac{\pi}{4}))^2\cdot(\sec\frac{\pi}{4})^2$

$=3(1)^2(\sqrt{2})^2$

$=6$

$\boxed{\begin{array}{l}\text{Tangent: } y-1=6(x-\frac{\pi}{4})\\ \text{Normal: } y-1=-\frac{1}{6}(x-\frac{\pi}{4})\end{array}}$

5. Use the limit definition of a derivative to evaluate the following derivatives. Check using basic differentiation rules.

a. $f(x)=\sqrt{3x-4}+4x$

$f'(x)=\frac{1}{2}(3x-4)^{-1/2}\cdot 3+4$

$=\frac{3}{2\sqrt{3x-4}}+4$

$\lim_{h\to 0}\frac{\sqrt{3x+3h-4}+4x+4h-\sqrt{3x-4}-4x}{h}=$

$\lim_{h\to 0}\left(\frac{\sqrt{3x+3h-4}-\sqrt{3x-4}}{h}+\frac{4h}{h}\right)=$

$\lim_{h\to 0}\left(\frac{3x+3h-4-3x+4}{h\sqrt{3x+3h-4}+\sqrt{3x-4}}\right)+4=$

$\lim_{h\to 0}\left(\frac{3}{\sqrt{3x+3h-4}+\sqrt{3x-4}}\right)+4=$

$\boxed{\frac{3}{2\sqrt{3x-4}}+4}$

b. $h(x)=5x+\frac{7}{x}$

$h'(x)=5-\frac{7}{x^2}$

$\lim_{h\to 0}\frac{5(x+h)+\frac{7}{x+h}-5x-\frac{7}{x}}{h}=$

$\lim_{h\to 0}\frac{\left(5h+\frac{7}{x+h}-\frac{7}{x}\right)x(x+h)}{(h)x(x+h)}=$

$\lim_{h\to 0}\frac{5xh(x+h)+7x-7x-7h}{h(x)(x+h)}=$

$\lim_{h\to 0}\frac{5x^2+5xh-7}{x(x+h)}=$

$\frac{5x^2-7}{x^2}=$

$\boxed{5-\frac{7}{x^2}}$

6. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

$y' = 6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$6(x-2)(x+1) = 0$

$x = 2 \quad x = -1$

$\boxed{\begin{array}{l}(2,0)\\(-1,27)\end{array}}$

7. Let $f(x) = \dfrac{\cos(x)}{\cos(x) - 2}$ for $-2\pi \le x \le 2\pi$.

a. Sketch a graph of f in the window $[-2\pi, 2\pi]$ by $[-2, 2]$.

b. Find $f'(x)$.

c. Find all values in the domain for which $f'(x) = 0$.

d. Use information obtained from parts (a) and (c) to find the range of f.

b. $f'(x) = \dfrac{-\sin x(\cos x - 2) + \sin x \cos x}{(\cos x - 2)^2} = \dfrac{2\sin x}{(\cos x - 2)^2}$

c. $f'(x) = 0 \Rightarrow 2\sin x = 0$

$\sin x = 0$

$x = 0, \pi, -2\pi, 2\pi, -\pi$

d. $[-1, \frac{1}{3}]$