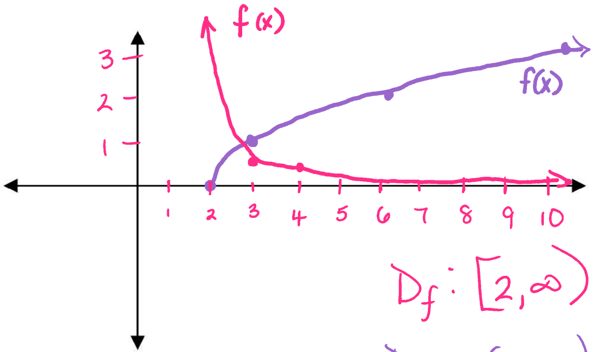
1. Given the $f(x)=\sqrt{x-2}$:
   a. Graph the function on the axes below.
   b. Find the derivative of $f(x)$ using the limit definition of a derivative.
   c. State the domain of $f(x)$ and $f'(x)$.
   d. Sketch $f'(x)$ on the same axis.
   e. What links do you observe between $f(x)$ and $f'(x)$?



$D_f: [2,\infty)$

$D_{f'}: (2,\infty)$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h-2}-\sqrt{x-2}}{h}\cdot\frac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{2}-(\cancel{x}-\cancel{2})}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$=\lim_{h\to 0}\frac{\cancel{h}^{\,1}}{\cancel{h}(\sqrt{x+h-2}+\sqrt{x-2})}$$

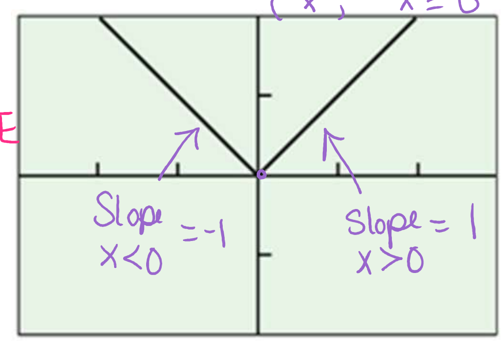
$$=\frac{1}{2\sqrt{x-2}}$$

2. Use the limit definition of the derivative at a point (i.e. $f'(a)=\frac{f(x)-f(a)}{x-a}$) to find, if possible $f'(0)$ for $f(x)=|x|$. (Hint: Find the limit from the left and from the right.)

$*\ f(x)=\begin{cases}-x, & x<0\\ x, & x\ge 0\end{cases}$

$$\left.\begin{aligned}\lim_{x\to 0^-}\frac{{}^{*}f(x)-0}{x-0}&=\lim_{x\to 0^-}\frac{-x}{x}=-1\\ \lim_{x\to 0^+}\frac{f(x)-0}{x-0}&=\lim_{x\to 0^+}\frac{x}{x}=1\end{aligned}\right\}\therefore \lim_{x\to 0}\frac{|x|-0}{x-0}\ \text{DNE}$$

$\boxed{f(x)=|x| \text{ is Non-Differentiable at } x=0}$

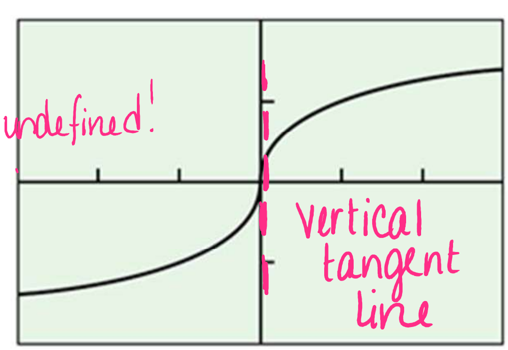


[−3, 3] by [−2, 2]

3. The function $g(x)=x^{\frac{1}{3}}$ is continuous at $x=0$. Find, if possible, $g'(0)$. (Hint: Use the limit definiti a point given in 1. above and evaluate from the left and the right.)

$$\left.\begin{aligned}\lim_{x\to 0^-}\frac{g(x)-0}{x-0}&=\lim_{x\to 0^-}\frac{x^{\frac{1}{3}}}{x}=\lim_{x\to 0^-}\frac{1}{x^{2/3}}=\infty\\ \lim_{x\to 0^+}\frac{g(x)-0}{x-0}&=\lim_{x\to 0^+}\frac{x^{\frac{1}{3}}}{x}=\lim_{x\to 0^+}\frac{1}{x^{2/3}}=\infty\end{aligned}\right\}g'(0)\text{ undefined!}$$

$\boxed{g(x)=x^{\frac{1}{3}} \text{ is Nondifferentiable at } x=0!}$



[−3, 3] by [−2, 2]

4. The function $h(x) = x^{\frac{2}{3}}$ is continuous at $x = 0$. Find, if possible, $h'(0)$. (Hint: Use the limit definiti a point given in 1. above and evaluate from the left and the right.)

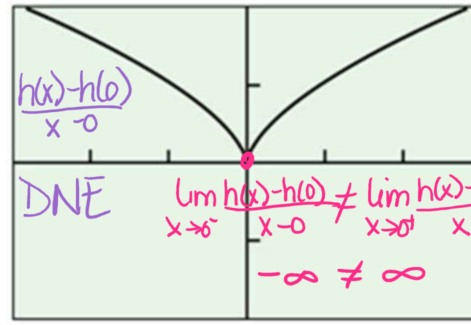
$$\lim_{x \to 0^-} \frac{h(x)-h(0)}{x-0} = \lim_{x \to 0^-} \frac{x^{2/3}}{x} = \lim_{x \to 0^-} \frac{1}{x^{1/3}} = -\infty$$

$$\lim_{x \to 0^+} \frac{h(x)-h(0)}{x-0} = \lim_{x \to 0^+} \frac{x^{2/3}}{x} = \lim_{x \to 0^+} \frac{1}{x^{1/3}} = \infty$$

$\lim_{x \to 0} \frac{h(x)-h(0)}{x-0}$ DNE

$\lim_{x \to 0^-} \frac{h(x)-h(0)}{x-0} \neq \lim_{x \to 0^+} \frac{h(x)-h(0)}{x-0}$

$-\infty \neq \infty$



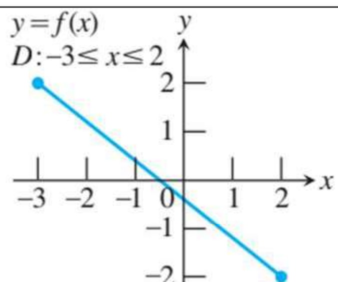
[-3, 3] by [-2, 2]

$\boxed{h(x) = x^{2/3} \text{ is Nondifferentiable at } x = 0.}$

5. The graph of a function over a closed interval D is given. On what domain does the function appear to be:

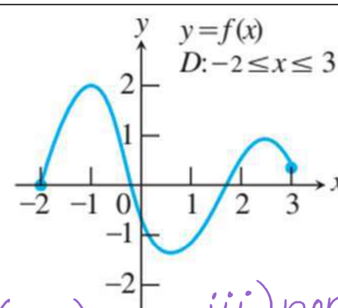
i. Differentiable?
ii. Continuous but not differentiable?
iii. Neither continuous nor differentiable?

a)



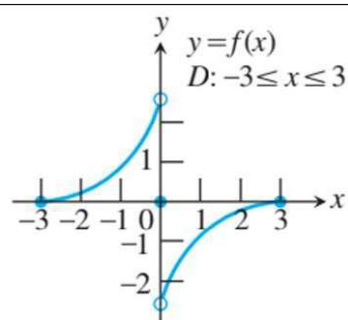
i) $(-3, 2)$
ii) $x = -3, 2$
iii) none

b)



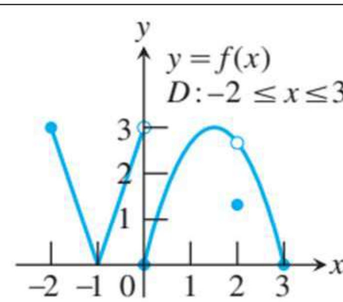
i) $(-2, 3)$
ii) $x = -2, 3$
iii) none

c)



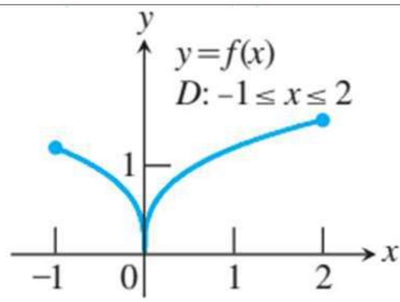
i) $(-3, 0) \cup (0, 3)$
ii) $x = -3, 3$
iii) $x = 0$

d)



i) $(-2, -1) \cup (-1, 0) \cup (0, 2) \cup (2, 3)$
ii) $x = -2, -1, 3$
iii) $x = 0, 2$

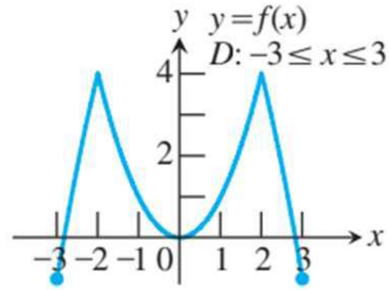
e)



i) $(-1,0)\cup(0,2)$

ii) $x=-1,0,2$ iii) none

f)



i) $(-3,-2)\cup(-2,2)\cup(2,3)$

ii) $x=\pm 3, \pm 2$ iii) none

6. Use the limit definition of the derivative at a point to justify why the given function is not differentiable at $x=0$. Then classify the point as a cusp, sharp turn or vertical tangent line.

a. $f(x)=x+\sqrt{x^2}+2$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x+\sqrt{x^2}+2-2}{x}$$

$$=\lim_{x\to 0}\frac{x+|x|}{x}=\lim_{x\to 0}\left(1+\frac{|x|}{x}\right)$$

$$\lim_{x\to 0^-}\left(1+\frac{|x|}{x}\right)=\lim_{x\to 0^-}\left(1+\frac{-x}{x}\right)=0$$

$$\lim_{x\to 0^+}\left(1+\frac{|x|}{x}\right)=\lim_{x\to 0^+}\left(1+\frac{x}{x}\right)=2$$

Sharp Turn

$$\lim_{x\to 0^-}f(x)=0\neq\lim_{x\to 0^+}f(x)=2$$

b. $g(x)=x^{\frac{4}{5}}$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{4/5}-0}{x}$$

$$=\lim_{x\to 0}\frac{1}{x^{1/5}}$$

$$\left.\begin{aligned}\lim_{x\to 0^-}\frac{1}{x^{1/5}}&=-\infty\\ \lim_{x\to 0^+}\frac{1}{x^{1/5}}&=\infty\end{aligned}\right\}\ \lim_{x\to 0}f(x)\ \text{DNE}$$

Cusp

c. $h(x)=3-\sqrt[3]{x}$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{3-\sqrt[3]{x}-(3-0)}{x}$$

$$=\lim_{x\to 0}\frac{-x^{1/3}}{x}=\lim_{x\to 0}\frac{-1}{x^{2/3}}$$

$$\left.\begin{aligned}\lim_{x\to 0^-}-\frac{1}{x^{2/3}}&=-\infty\\ \lim_{x\to 0^+}-\frac{1}{x^{2/3}}&=-\infty\end{aligned}\right\}\ \lim_{x\to 0^+}-\frac{1}{x^{1/3}}=-\infty$$

$\therefore h'(0)$ undefined

Vertical Tan Line

7. Given the following functions, write the equation of the tangent line at the given point:

a. $f(x)=x^2+2x+1;\quad(-3,4)$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2+2(x+h)+1-(x^2+2x+1)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2+\cancel{2x}+2h\cancel{+1}\cancel{-x^2}\cancel{-2x}\cancel{-1}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+2h+1)}{\cancel{h}}=2x+2$$

$(-3,4)\quad m=f'(-3)=2(-3)+2=-4$

$$\boxed{y-4=-4(x+3)}$$

b. $h(x)=\sqrt{x-1};\quad(5,2)$

$$h'(x)=\lim_{h\to 0}\frac{\sqrt{x+h-1}-\sqrt{x-1}}{h}\cdot\frac{\sqrt{x+h-1}+\sqrt{x-1}}{\sqrt{x+h-1}+\sqrt{x-1}}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h\cancel{-1}-(\cancel{x}\cancel{-1})}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}=\frac{1}{2\sqrt{x-1}}$$

$(5,2)\quad m=h'(5)=\frac{1}{2\sqrt{5-1}}=\frac{1}{4}$

$$\boxed{y-2=\frac{1}{4}(x-5)}$$

c. $g(x)=3x-\dfrac{2}{x};\quad(1,1)$

$$g'(x)=\lim_{h\to 0}\frac{3(x+h)-\frac{2}{x+h}-\left(3x-\frac{2}{x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{3x}+3h-\cancel{3x}}{h}+\lim_{h\to 0}\frac{\left(\frac{-2}{x+h}+\frac{2}{x}\right)x(x+h)}{h(x(x+h))}$$

$$=\lim_{h\to 0}\frac{3\cancel{h}}{\cancel{h}}+\lim_{h\to 0}\frac{-\cancel{2x}+\cancel{2x}+2h}{h(x(x+h))}$$

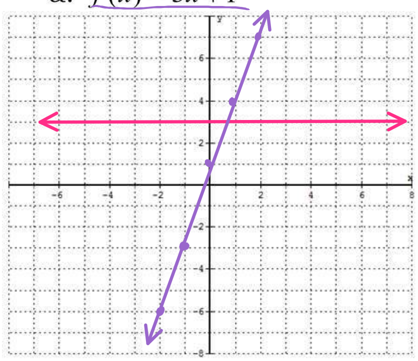
$$=\lim_{h\to 0}3+\lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}(x(x+h))}=3+\frac{2}{x^2}$$

$(1,1)\quad m=g'(1)=3+\frac{2}{1}=5$

$$\boxed{y-1=5(x-1)}$$

8. Find the derivative of the following functions using the limit process. Then, sketch the func and its derivative on the same axis. Any observations?

a. $f(x)=3x+1$



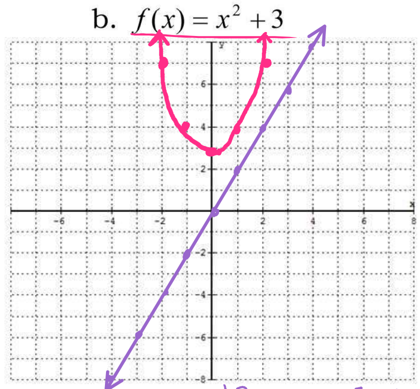
$f'(x)=\lim\limits_{h\to0}\dfrac{3(x+h)+1-(3x+1)}{h}$

$=\lim\limits_{h\to0}\dfrac{\cancel{3x}+3h+\cancel{1}-\cancel{3x}-\cancel{1}}{h}$

$=\lim\limits_{h\to0}\dfrac{3h}{h}=3$

$f'(x)=3$

b. $f(x)=x^2+3$



$f'(x)=\lim\limits_{h\to0}\dfrac{(x+h)^2+3-(x^2+3)}{h}$

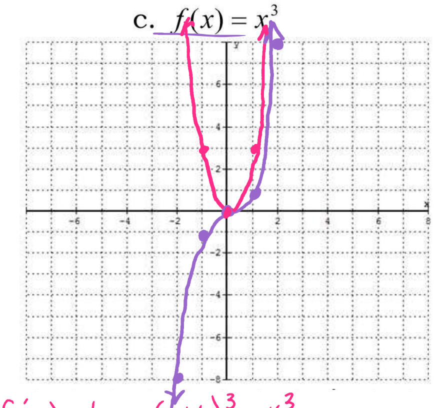
$=\lim\limits_{h\to0}\dfrac{\cancel{x^2}+2xh+h^2+\cancel{3}-\cancel{x^2}-\cancel{3}}{h}$

$=\lim\limits_{h\to0}\dfrac{\cancel{h}(2x+h)}{\cancel{h}}=\lim\limits_{h\to0}(2x+h)$

$=2x$

$f'(x)=2x$

c. $f(x)=x^3$



$f'(x)=\lim\limits_{h\to0}\dfrac{(x+h)^3-x^3}{h}$

$=\lim\limits_{h\to0}\dfrac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$

$=\lim\limits_{h\to0}\dfrac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}$

$=\lim\limits_{h\to0}(3x^2+3xh+h^2)=3x^2$

$f'(x)=3x^2$

9. Identify any points at which the function is continuous but not differentiable. Explain why.

a. $f(x)=\dfrac{4}{3x-2}$

VA at $x=2/3$

Continuous & differentiable on

$(-\infty,2/3)\cup(2/3,\infty)$

Nonremovable, infinite discont. at $x=2/3$.

b. $f(x)=\sqrt{x-1}$

Continuous on $[1,\infty)$

Non differentiable at $x=1$

$\lim\limits_{x\to1^+}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to1^+}\dfrac{\sqrt{x-1}-\sqrt{0}}{x-1}$

$=\lim\limits_{x\to1^+}\dfrac{(x-1)^{1/2}}{x-1}=\lim\limits_{x\to1^+}\dfrac{1}{(x-1)^{1/2}}=\infty$

Vert tan line at $x=1$

c. $f(x)=\sqrt[3]{x+2}$

Continuous on $(-\infty,\infty)$

Non differentiable at $x=-2$

$\lim\limits_{x\to-2}\dfrac{(x+2)^{1/3}-0}{x+2}=\lim\limits_{x\to-2}\dfrac{1}{(x+2)^{2/3}}$

$\lim\limits_{x\to-2^-}\dfrac{1}{(x+2)^{2/3}}=\infty=\lim\limits_{x\to-2^+}\dfrac{1}{(x+2)^{2/3}}$

$\therefore f'(2)=\lim\limits_{x\to-2}\dfrac{1}{(x+2)^{2/3}}=\infty$ undefined

Vertical tan line at $x=-2$.

d. $f(x)=-|x|$

Continuous on $(-\infty,\infty)$

Nondifferentiable at $x=0$

$\lim\limits_{x\to0}\dfrac{-|x|-0}{x-0}=\lim\limits_{x\to0}\dfrac{-|x|}{x}$

$\left.\begin{aligned}&\lim\limits_{x\to0^-}\dfrac{-|x|}{x}=\lim\limits_{x\to0^-}\dfrac{-(-x)}{x}=1\\&\lim\limits_{x\to0^+}\dfrac{-|x|}{x}=\lim\limits_{x\to0^+}\dfrac{-x}{x}=-1\end{aligned}\right\}\ \lim\limits_{x\to0}\dfrac{-|x|}{x}$ DNE

Sharp Turn