Name Answer Key   Date________________

Calc I H – 3.7 Review   **Period** ____________

1. What positive number *x* minimizes the sum of *x* and its reciprocal?

$S = x + \frac{1}{x}$

$S' = 1 - \frac{1}{x^2} = 0$

$\frac{1}{x^2} = 1$

$x^2 = 1$

$x = \pm 1$ $\quad x \neq -1$

$x = 1$

$S'' = \frac{2}{x^3}$

$S''(1) = 2 > 0$ and $S'(1) = 0$

$\boxed{\text{Sum is a min when } x = 1.}$

2. The sum of one number and 3 times a second number is 36. What numbers should be selected so that their product is as large as possible?

$P = xy \qquad x + 3y = 36$

$x = 36 - 3y$

$P = (36 - 3y)y$

$P = 36y - 3y^2$

$P' = 36 - 6y$

$P' = 36 - 6y = 0$

$36 = 6y$

$y = 6$

$P'' = -6$

$P''(6) < 0$ & $P'(6) = 0$

Product is max when $y = 6$.

$x = 36 - 3(6) = 18$

$\boxed{\text{Numbers: } 18 \text{ \& } 6}$

3. Find two positive numbers whose sum is 110 and product is a maximum.

$P = xy$

$P = x(110 - x)$

$P = 110x - x^2$

$P' = 110 - 2x$

$x + y = 110$

$\Rightarrow y = 110 - x$

$P' = 110 - 2x = 0$

$2x = 110$

$x = 55$

$P'' = -2 < 0$ & $P'(55) = 0$

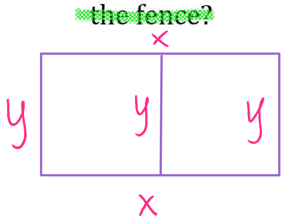
Product is max when $x = 55$.

$55 + y = 110$

$y = 55$

$\boxed{\text{Numbers: } 55 \text{ \& } 55}$

4. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?



$xy = 1.5$

$x = \frac{1.5}{y}$

$F = 2x + 3y$

$F = 2\left(\frac{1.5}{y}\right) + 3y$

$F = \frac{3}{y} + 3y$

$F' = \frac{-3}{y^2} + 3 = 0$

$3 = \frac{3}{y^2}$

$3y^2 = 3$

$y^2 = 1{,}000{,}000$

$y = \pm 1000$

$y \neq -1000$

$y = 1000$ ft

$x = \frac{1{,}500{,}000}{1000}$

$x = 1500$ ft

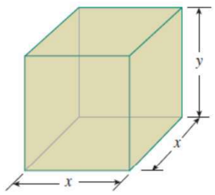
$F'' = \frac{6}{y^3}$

$F''(1) = 6 > 0$ & $F'(1) = 0$

Cost of fence is minimum when $y = 1000$ ft

$\boxed{1500 \text{ ft} \times 1000 \text{ ft}}$

5. A rectangular box with a square base and open top must have a volume of 32,000 cm³. What are the dimensions of the box that minimize the amount of material used? → min surface area



$V = x^2 y = 32{,}000$

$y = \frac{32{,}000}{x^2}$

$A = 4xy + x^2$

$A = \frac{128{,}000}{x} + x^2$

$A' = -\frac{128{,}000}{x^2} + 2x = 0$

$2x = \frac{128{,}000}{x^2}$

$x^3 = 64{,}000$

$x = 40$

$y = \frac{32{,}000}{(40)^2}$

$y = 20$ cm

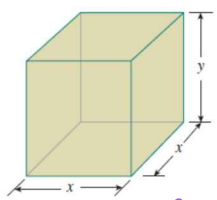
$A'' = \frac{256{,}000}{x^3} + 2$

$A''(4000) > 0$ & $A'(4) = 0$

min surface area occurs when $x = 40$ cm

$\boxed{40\text{ cm} \times 40\text{ cm} \times 20\text{ cm}}$

6. If 1200 cm² of material is available to make a rectangular box with a square base and an open top, find the largest possible volume of the box.



$1200 = 4xy + x^2$

$y = \frac{1200 - x^2}{4x}$

$V = x^2 y$

$V = x^{\cancel{2}}\left(\frac{1200 - x^2}{4\cancel{x}}\right)$

$V = 300x - \frac{x^3}{4}$

$V' = 300 - \frac{3x^2}{4} = 0$

$\frac{3x^2}{4} = 300$

$x^2 = 400$

$x = \pm 20$

$x \neq -20$

$x = 20$

$V'' = -\frac{6x}{4} = \frac{-3x}{2}$

$V''(20) = -30 < 0$, $V'(20) = 0$

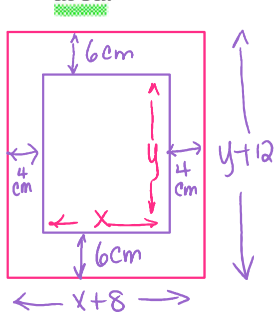
Volume is a min when $x = 20$ cm

$y = \frac{1200 - 400}{80}$

$y = 10$ cm

$\boxed{20\text{cm} \times 20\text{cm} \times 10\text{cm} \quad V = 4000\text{ cm}^3}$

7. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of the printed material on the poster is fixed at 384 cm², find the dimensions of the poster with the smallest area.



$xy = 384 \Rightarrow x = \frac{384}{y}$

$A = (x+8)(y+12)$

$A = \left(\frac{384}{y} + 8\right)(y+12)$

$A = 384 + \frac{4608}{y} + 8y + 96$

$A = 480 + \frac{4608}{y} + 8y$

$A' = -\frac{4608}{y^2} + 8 = 0$

$8y^2 = 4608$

$y^2 = 576$

$y = \pm 24$

$y \neq -24$

$y = 24$, $x = 16$

$+12$ $\quad +8$

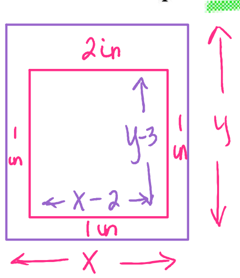
$A'' = \frac{9216}{y^3}$

$A''(24) > 0$ & $A'(24) = 0$

Area is min when $y = 24$ cm.

$\boxed{24\text{ cm} \times 36\text{ cm}}$

8. A poster is to have an area of 180 in² with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What overall poster dimensions will maximize the space for the poster's image?



$xy = 180$

$y = \frac{180}{x}$

$A = (x-2)(y-3)$

$A = (x-2)\left(\frac{180}{x} - 3\right)$

$A = 180 - 3x - \frac{360}{x} + 6$

$A = 186 - 3x - \frac{360}{x}$

$A' = -3 + \frac{360}{x^2} = 0$

$3x^2 = 360$

$x^2 = 120$

$x = \pm 2\sqrt{30}$

$x \neq -2\sqrt{30}$

$x = 2\sqrt{30}$

$y = \frac{180}{2\sqrt{30}} = \frac{90\sqrt{30}}{30} = 3\sqrt{30}$

$A'' = \frac{-720}{x^3}$

$A''(2\sqrt{30}) < 0$ & $A'(2\sqrt{30}) = 0$

Image area is max when $x = 2\sqrt{30}$ in.

$\boxed{2\sqrt{30}\text{ in} \times 3\sqrt{30}\text{ in}}$

*Challenge An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

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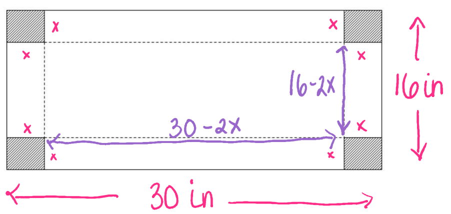
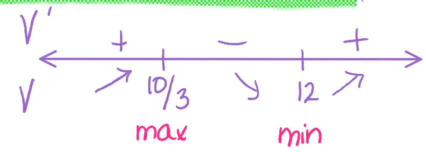


$V = x(16-2x)(30-2x)$

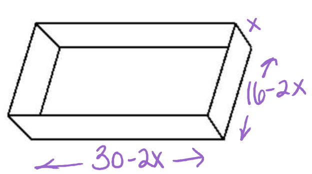
$V = x(480 - 92x + 4x^2)$

$V = 480x - 92x^2 + 4x^3$

$V' = \dfrac{480 - 184x + 12x^2}{4} = \dfrac{0}{4}$



Volume of box will be a max when $x = 10/3$ since V incr → decr.



360, −36, −10, −46

$3x^2 - 46x + 120 = 0$

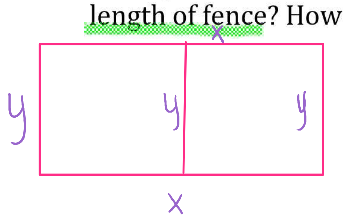
$(3x^2 - 36x)(-10x + 120) = 0$

$3x(x-12) - 10(x-12) = 0$

$(3x-10)(x-12) = 0 \Rightarrow x = 10/3, 12$

$\boxed{\text{Squares } 3\tfrac{1}{3} \text{ in} \times 3\tfrac{1}{3} \text{ in}}$

10. A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?



$xy = 216$

$y = \dfrac{216}{x}$

$f = 3y + 2x$

$f = 3\left(\dfrac{216}{x}\right) + 2x$

$f = \dfrac{648}{x} + 2x$

$f' = -\dfrac{648}{x^2} + 2 = 0$

$2x^2 = 648$

$x^2 = 324$

$x = \pm 18$ $\quad x \neq -18$

$x = 18, \quad y = \dfrac{216}{18} = 12$

$f'' = \dfrac{1296}{x^3}$

$f''(18) > 0$ & $f'(18) = 0$

fence dimensions are min when $x = 18$ m.

$\boxed{18\text{m} \times 12\text{m}}$

$f = 3(12) + 2(18)$

$= 36 + 36$

$= 72$

$\boxed{\text{72 m of fence will be needed.}}$

11. What two nonnegative real numbers with a ~~product~~ sum of 23 have the ~~smallest~~ largest possible ~~sum~~ product?

$x + y = 23$

$y = 23 - x$

$P = xy$

$P = x(23 - x)$

$P = 23x - x^2$

$P' = 23 - 2x = 0$

$x = \dfrac{23}{2}$

$P'' = -2, \; P'\left(\dfrac{23}{2}\right) = 0$

$\therefore$ Product is a max when $x = \dfrac{23}{2}$

$y = \dfrac{23}{2}$

$\boxed{\text{Numbers: } \dfrac{23}{2}, \dfrac{23}{2}}$

12. Find numbers a and b such that their product is as large as possible and the sum of three times a and b is twelve.

$3a + b = 12$

$b = 12 - 3a$

$P = ab$

$P = a(12 - 3a)$

$P = 12a - 3a^2$

$P' = 12 - 6a = 0$

$6a = 12$

$a = 2$

$P'' = -4$ & $P'(2) = 0$

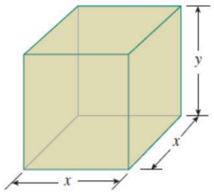
Product is max when $a = 2$

$b = 12 - 3(2) = 6$

$\boxed{\text{Numbers: } a = 2, \; b = 6}$

13. Find the dimensions of a box with a square base that has a:

a. volume of 12 ft³ and minimal surface area.



$x^2 y = 12$

$y = \frac{12}{x^2}$

$A = 2x^2 + 4xy$

$A = 2x^2 + 4x\left(\frac{8}{x^2}\right)$

$A = 2x^2 + \frac{32}{x}$

$A' = 4x - \frac{32}{x^2} = 0$

$4x = \frac{32}{x^2}$

$x^3 = 12$

$x = \sqrt[3]{12}$

$y = \frac{12}{(\sqrt[3]{12})^2}$

$y = \frac{12}{12^{2/3}}$

$y = 12^{1/3}$

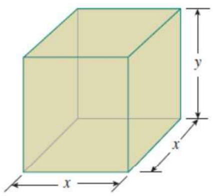
$A'' = 4 + \frac{64}{x^3}$

$A''(2) > 0$ & $A'(2) = 0$

Surface area is min when $x = \sqrt[3]{12}$ ft

$\boxed{12^{1/3}\text{ft} \times 12^{1/3}\text{ft} \times 12^{1/3}\text{ft}}$

b. surface area of 20 cm² and maximal volume.



$2x^2 + 4xy = 20$

$y = \frac{20 - 2x^2}{4x}$

$V = x^2 y$

$V = x^{\cancel{2}}\left(\frac{20 - 2x^2}{4\cancel{x}}\right)$

$V = 5x - \frac{x^3}{2}$

$V' = 5 - \frac{3x^2}{2} = 0$

$\frac{3x^2}{2} = 5$

$3x^2 = 10$

$x^2 = 10/3$

$x = \pm\sqrt{10/3}$

$x \neq -\sqrt{10/3}$

$x = \sqrt{10/3}$

$y = \frac{20 - 2(10/3)}{4\sqrt{\frac{10}{3}}} = \frac{\frac{60-20}{3}}{4\sqrt{\frac{10}{3}}} = \sqrt{\frac{10}{3}}$

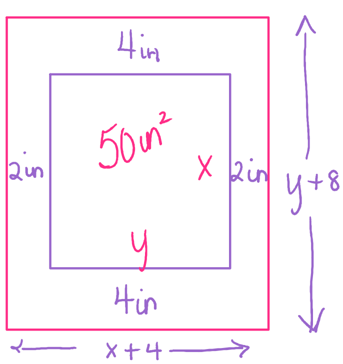
$V'' = -3x$

$V''(4) = -12 < 0$, $V'\left(\sqrt{\frac{10}{3}}\right) = 0$

Volume is max when $x = \sqrt{\frac{10}{3}}$ cm

$\boxed{\sqrt{\frac{10}{3}}\text{cm} \times \sqrt{\frac{10}{3}}\text{cm} \times \sqrt{\frac{10}{3}}\text{cm}}$

14. You are designing a rectangular poster to contain 50 in² of printing with a 4 inch margin at the top and bottom and a 2 inch margin at each side. What overall dimensions will minimize the amount of paper used?



$xy = 50$

$y = \frac{50}{x}$

$A = (x+4)(y+8)$

$A = (x+4)\left(\frac{50}{x} + 8\right)$

$A = 50 + 8x + \frac{200}{x} + 32$

$A = 82 + 8x + \frac{200}{x}$

$A' = 8 - \frac{200}{x^2} = 0$

$8x^2 = 200$

$x^2 = 25$

$x = \pm 5$

$x \neq -5$

$x = 5$

$A'' = \frac{400}{x^3}$

$A''(5) > 0$ & $A'(5) = 0$

Area is min when $x = 5$ in.

$y = \frac{50}{5} = 10$ in

$w = x + 4$ $\quad w = 9$ in

$l = y + 8$ $\quad l = 18$ in

$\boxed{9\text{in} \times 18\text{in}}$