AP Calculus AB **Section 4.1:** Antiderivatives and Particular Solutions **Day 2**

We saw yesterday that a differential equation has many solutions, each differing from one another by a constant *c*. This means the graph of any two antiderivatives of *f* are vertical translations of each other.

In many applications of integration, you are given enough information to determine a **particular solution**. By being given the **initial condition** of the function, you have enough information to find the value of *c*.

Examples:

1. Find the particular solution to the differential equation given $f'(x) = x^2 - 2x + 2$ and $f(3) = -1$.

$\frac{dy}{dx} = x^2 - 2x + 2$

$\int dy = \int (x^2 - 2x + 2)\,dx$

$y = \frac{x^3}{3} - x^2 + 2x + C$

$-1 = \frac{(3)^3}{3} - 3^2 + 2(3) + C$

$-1 = 6 + C \quad C = -7$

$\boxed{f(x) = \frac{x^3}{3} - x^2 + 2x - 7}$

2. Solve the differential equation given $f''(x) = 2$, $f'(4) = 1$ and $f(-1) = 2$.

$\int f''(x)\,dx = \int 2\,dx$

$f'(x) = 2x + C$

$f'(4) = 2(4) + C = 1$

$C = -7$

$f'(x) = 2x - 7$

$\int f'(x)\,dx = \int (2x - 7)\,dx$

$f(x) = x^2 - 7x + C$

$f(-1) = (-1)^2 - 7(-1) + C = 2$

$8 + C = 2$

$C = -6$

$\boxed{f(x) = x^2 - 7x - 6}$

3. A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet. Find the position function giving the height *s* as a function of time *t*. When does the ball hit the ground?

$f(0) = 80$

$f'(0) = 64$

$f''(x) = -32 \text{ ft/sec}^2$

$\int f''(x)\,dx = \int -32\,dx$

$f'(x) = -32x + C$

$f'(0) = C = 64$

$f'(x) = -32x + 64$

$\int f'(x)\,dx = \int (-32x + 64)\,dx$

$f(x) = -16x^2 + 64x + C$

$f(0) = 80 = C$

$\boxed{f(x) = -16x^2 + 64x + 80}$

$-16x^2 + 64x + 80 = 0$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5 \quad x = 1$

$\boxed{x = 5 \text{ sec.}}$

4. Solve the differential equation given $f''(x) = 2x$, $f'(-5) = 30$ and $f(2) = -1$.

$\int f''(x)dx = \int 2x\,dx$

$f'(x) = x^2 + C$

$f'(-5) = (-5)^2 + C = 30$

$C = 5$

$f'(x) = x^2 + 5$

$\int f'(x)\,dx = \int (x^2+5)\,dx$

$f(x) = \frac{x^3}{3} + 5x + C$

$f(2) = \frac{2^3}{3} + 5(2) + C = -1$

$\frac{8}{3} + 10 + C = -1$

$\frac{38}{3} + C = -1$

$C = \frac{-41}{3}$

$\boxed{f(x) = \frac{x^3}{3} + 5x - \frac{41}{3}}$

5. Given that the graph of $f(x)$ passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, find $f(6)$.

$f'(x) = 2x + 1$

$\int f'(x)\,dx = \int (2x+1)\,dx$

$f(x) = x^2 + x + C$

$f(1) = 1^2 + 1 + C = 6$

$C = 4$

$f(x) = x^2 + x + 4$

$f(6) = 6^2 + 6 + 4 = \boxed{46}$

6. Solve the differential equation given $f''(x) = \frac{1}{x^{3/2}}$, $f'(4) = 2$ and $f(0) = 1$.

$\int f''(x)\,dx = \int x^{-3/2}\,dx$

$f'(x) = -2x^{-1/2} + C$

$f'(4) = \frac{-2}{\sqrt{4}} + C = 2$

$C = 3$

$f'(x) = \frac{-2}{\sqrt{x}} + 3$

$\int f'(x)\,dx = \int (-2x^{-1/2} + 3)\,dx$

$f(x) = -4\sqrt{x} + 3x + C$

$f(0) = C = 1$

$\boxed{f(x) = -4\sqrt{x} + 3x + 1}$

7. Solve the differential equation given $f''(x) = \cos(x)$, $f'(\pi) = 2$ and $f(\pi) = -1$.

$\int f''(x)\,dx = \int \cos x\,dx$

$f'(x) = \sin x + C$

$f'(\pi) = \sin\pi + C = 2$

$C = 2$

$f'(x) = \sin x + 2$

$\int f'(x)\,dx = \int (\sin x + 2)\,dx$

$f(x) = -\cos x + 2x + C$

$f(\pi) = -\cos\pi + 2\pi + C = -1$

$C = -2 - 2\pi$

$\boxed{f(x) = -\cos x + 2x - 2 - 2\pi}$

8. Find the general solution to each indefinite integral.

a. $\int\left(\frac{2}{3}x^5-\frac{5}{2}x+\frac{1}{2}\right)dx$

$= \boxed{\frac{x^6}{9}-\frac{5x^2}{4}+\frac{x}{2}+C}$

b. $\int\frac{2}{\sqrt{1-x^2}}dx$

$= \boxed{2\sin^{-1}(x)+C}$

c. $\int 4(5x-3)^2\,dx$

$= 4\int(25x^2-30x+9)\,dx$

$= 4\left(\frac{25x^3}{3}-15x^2+9x+C\right)$

$= \boxed{\frac{100x^3}{3}-60x^2+36x+C}$

d. $\int\frac{x^4-3x+5}{x^2}dx$

$= \int(x^2-3x^{-1}+5x^{-2})\,dx$

$= \boxed{\frac{x^3}{3}-3\ln x-\frac{5}{x}+C}$

e. $\int\frac{(3x-2)^2}{\sqrt{x}}dx$

$= \int\frac{9x^2-12x+4}{x^{1/2}}dx$

$= \int(9x^{3/2}-12x^{1/2}+4x^{-1/2})\,dx$

$= \boxed{\frac{18}{5}x^{5/2}-8x^{3/2}+8x^{\frac{1}{2}}+C}$

f. $\int\left(e^x-\frac{1}{x^{\frac{3}{4}}}\right)dx$

$= \boxed{e^x-4x^{1/4}+C}$

g. $\int\left(\frac{1}{x^2}-\sin x\right)dx$

$= -x^{-1}+\cos x+C$

$= \boxed{-\frac{1}{x}+\cos x+C}$

h. $\int\left(\frac{\sin x}{1-\sin^2 x}\right)dx$

$= \int\frac{\sin x}{\cos^2 x}dx$

$= \int\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}dx$

$= \int\tan x\sec x\,dx$

$= \boxed{\sec x+C}$

i. $\int(\sin^2 x+\cos^2 x)\,dx$

$= \int 1\,dx$

$= \boxed{x+C}$