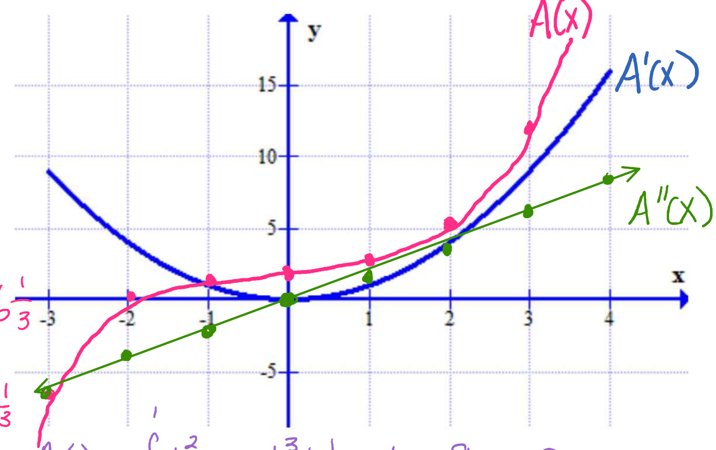
AP Calc AB **Section 4.4:** The Second Fundamental Theorem of Calculus **(Days 3 & 4)**

1. The function below is $f(x)=x^2$. The accumulation function $A(x)$ is the amount of "accumulated **signed** area" from –2 to x or $A(x)=\int_{-2}^{x} t^2\,dt$.



a. On the graph, let x vary from –3 to 3. Find the "accumulation" for each integer x-value on the graph.

$A(-3)=\int_{-2}^{-3} t^2 dt = -\int_{-3}^{-2} t^2 dt = -\frac{t^3}{3}\Big|_{-3}^{-2} = \left(\frac{-8}{3}+\frac{27}{3}\right)$

$= -\frac{19}{3} = -6\frac{1}{3}$

$A(-2)=\int_{-2}^{-2} t^2 dt = 0$

$A(-1)=\int_{-2}^{-1} t^2 dt = \frac{t^3}{3}\Big|_{-2}^{-1} = \frac{-1}{3}+\frac{8}{3} = \frac{7}{3} = 2\frac{1}{3}$

$A(0)=\int_{-2}^{0} t^2 dt = \frac{t^3}{3}\Big|_{-2}^{0} = 0+\frac{8}{3} = \frac{8}{3} = 2\frac{2}{3}$

$A(1)=\int_{-2}^{1} t^2 dt = \frac{t^3}{3}\Big|_{-2}^{1} = \frac{1}{3}+\frac{8}{3} = 3$

$A(2)=\int_{-2}^{2} t^2 dt = \frac{t^3}{3}\Big|_{-2}^{2} = \frac{8}{3}+\frac{8}{3} = \frac{16}{3} = 5\frac{1}{3}$

$A(3)=\int_{-2}^{3} t^2 dt = \frac{t^3}{3}\Big|_{-2}^{3} = \frac{27}{3}+\frac{8}{3} = \frac{35}{3} = 11\frac{2}{3}$

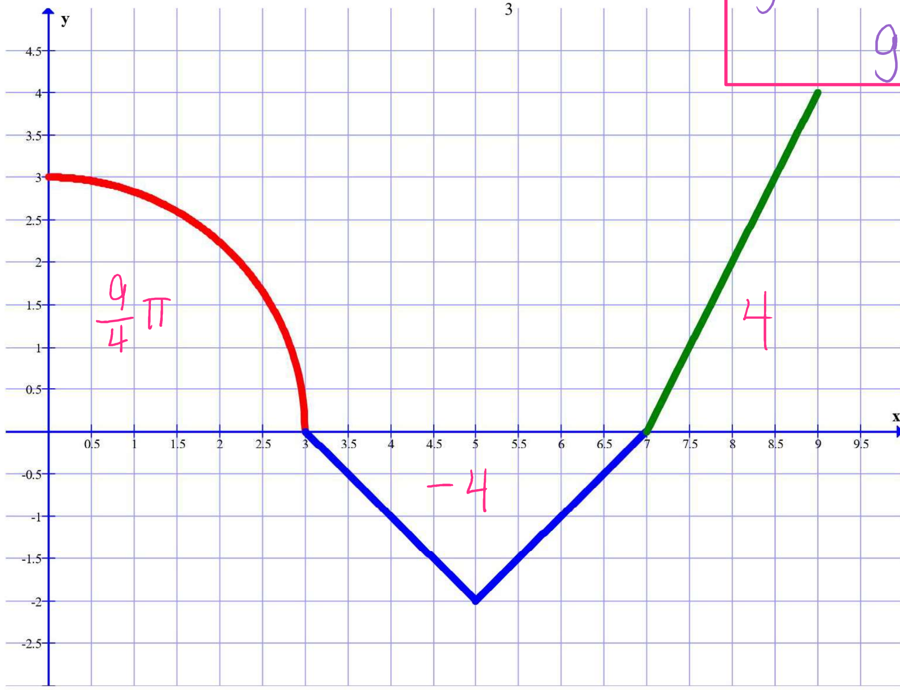
b. What are the values of $A'(x)$? What are the values of $A''(x)$?

$A'(x) = \frac{d}{dx}\int_{-2}^{x} t^2 dt = x^2 = f(x)$

$A''(x) = f'(x)$

$= 2x$ (graphed above)

2. Let $f(x)$ be a function defined in the closed interval $0 \le x \le 9$. The graph of $f(x)$ consists of a quarter circle and three line segments. Let $g(x) = 4 + \int_3^x f(t)\,dt$. $\Rightarrow$ $\boxed{g'(x) = f(x) \quad g''(x) = f'(x)}$



a. Find $g(4)$, $g'(4)$, $g''(4)$.

$g(4) = 4 + \int_3^4 f(t)\,dt = 4 - \frac{1}{2} = 3\frac{1}{2}$ $\quad g'(4) = f(4) = -1 \quad g''(4) = f'(4) = -1$

b. What is the average value of $f(x)$ on the interval $3 \le x \le 9$?

$\text{Av} f = \frac{1}{9-3} \int_3^9 f(x)\,dx = \frac{1}{6}(-4+4) = 0$

c. What is the average rate of change of $g(x)$ on the interval $3 \le x \le 9$?

$g(9) = 4 + \int_3^9 f(x)\,dx = 4$

$g(3) = 4 + \int_3^3 f(x)\,dx = 4$

$\frac{g(9) - g(3)}{9-3} = \frac{0}{6} = 0$

d. Identify the x-coordinate(s) of any extrema of $g(x)$ on $0 \le x \le 9$. Explain your reasoning.

$g'(x) = f(x)$ Rel max at $x = 3$, $g'(x) > 0 \to g'(x) < 0$ * See below

Rel min at $x = 7$, $g'(x) < 0 \to g'(x) > 0$

e. Identify the x-coordinate(s) of any point(s) of inflection on $0 \le x \le 9$.

$g''(x) = f'(x)$ POI at $x = 5$ since $g''(x) < 0 \to g''(x) > 0$.

or $g'(x)$ changes from decr $\to$ incr

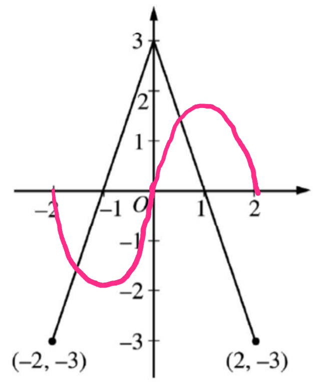
Must consider endpoints on Closed Interval:

$g(0) = 4 + \int_3^0 f(t)\,dt = 4 - \int_0^3 f(t)\,dt = 4 - \frac{9\pi}{4} < 0$ $\quad g(9) = 4 + \int_3^9 f(t)\,dt = 4$

$g(3) = 4 + \int_3^3 f(t)\,dt = 4$ $\quad g(7) = 4 + \int_3^7 f(t)\,dt = 4 - 4 = 0$

$\therefore$ Abs max $(3, 4)$, $(9, 4)$ Abs min $(0, 4 - \frac{9\pi}{4})$ Rel min $(7, 0)$

3.



Graph of f

$g(x) = \int_0^x f(t)\,dt$

$g'(x) = f(x)$

$g''(x) = f'(x)$

The graph of the function f shown above consists of two line segments.

Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

$g(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^{0} f(t)dt = -\frac{3}{2}$

a. Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

$g'(-1) = f(-1) = 0$

$g''(-1) = f'(-1) = 3$

b. For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

g increases when $g'(x) = f(x) > 0$. g increasing on $(-1, 1)$.

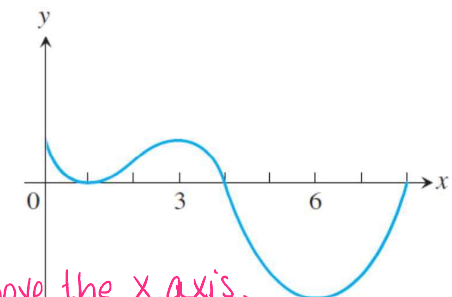
c. For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.

g concave down when $g''(x) < 0$ or $g'(x) = f(x)$ decr. g concave down on $(0, 2)$.

d. On the axes provided, sketch the graph of g on the closed interval $[-2, 2]$.

See above

4. The graph below is of a continuous function f with domain $[0, 8]$. Let h be the function defined by $h(x) = \int_1^x f(t)dt$



a. Find $h(1)$. $h(1) = \int_1^1 f(t)dt = 0$

b. Is $h(0)$ positive or negative? Justify your answer.

$h(0) = \int_1^0 f(t)dt = -\int_0^1 f(t)dt$. Since the area is above the x axis, $\int_0^1 f(t)dt > 0$. ∴ $h(0) < 0$.

c. Find the value of x for which $h(x)$ is a maximum.

$h(4)$ is a maximum. At $x = 4$, $h'(x) > 0 \rightarrow h'(x) < 0$.

$h'(x)$	+	+	−
$h(x)$	↗	↗	↘
0	1	4	8

d. Find the value of x for which $h(x)$ is a minimum.

$h(8)$ is a minimum. $h(8) = \int_1^8 f(t)dt = \int_1^4 f(t)dt + \int_4^8 f(t)dt$

Since $\int_1^4 f(t)dt < \int_4^8 f(t)dt$, $h(8) < 0$. OR since pos. area < neg. area, $h(8)$ is min.

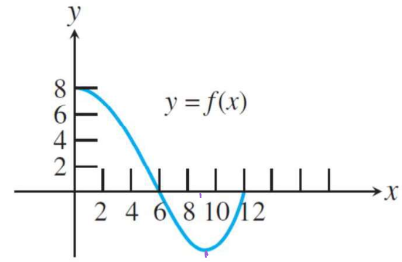
+ −

e. Find the x-coordinates of all points of inflection of the graph of $y = h(x)$.

$h'(x) = f(x)$; $h''(x) = f'(x)$ POI at $x = 1, 3, 6$ since $h''(x)$ changes sign (or $h'(x)$ changes incr ⇄ decr).

5. Let $H(x)=\int_0^x f(t)dt$, where f is continuous function with domain $[0, 12]$.

a. Find $H(0)$. $H(0)=\int_0^0 f(t)dt=0$



b. On what interval is H increasing? Justify.

$H'(x)=f(x)$ $H(x)$ increasing on $(0,6)$ since $H'(x)>0$.

c. One what interval is the graph of H concave up? Justify.

$H''(x)=f'(x)$ $H(x)$ concave up on $(9,12)$ since $H'(x)$ incr and $H''(x)>0$.

d. Is $H(12)$ $\boxed{\text{positive}}$ or negative? Justify.

$H(12)=\int_0^{12} f(t)dt=\int_0^6 f(t)dt+\int_6^{12} f(t)dt$. Since the area above the x axis $(\int_0^6 f(t)dt)$ > area below x axis $(\int_6^{12} f(t)dt)$. $H(12)>0$

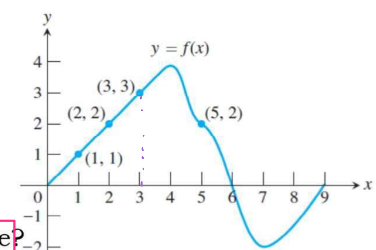
e. Where does H achieve its maximum value? Justify.

$H(x)$ is max at $x=6$ since $H'(x)>0 \rightarrow H'(x)<0$ (completely incr → completely decr at $x=6$.)

f. Where does H achieve its minimum value? Justify.

$H(x)$ is min at $x=0$. Since $H(12)=\int_0^{12} f(t)dt>0$ (see d), $H(12)>H(0)$. and $H(0)=0$

6. In the following example f is a differentiable function whose graph is given below. The position at time t (sec) of a particle moving along a coordinate axis is $s=\int_0^t f(x)dx$ meters. Use the graph to answer the questions and give reasons for your answers.



a. What is the particle's velocity at $t=5$?

$v(t)=s'(t)=f(t)$ $v(5)=2$ m/s

b. Is the acceleration of the particle at time $t=5$ positive or $\boxed{\text{negative}}$?

$a(t)=f'(t)$ $a(5)<0$ since $f(t)$ is decreasing at $t=5$.

c. What is the particle's position at time $t=3$?

$s(3)=\int_0^3 f(t)dt=\frac{9}{2}$

d. At what time during the first 9 seconds does s have its largest value?

$s(t)$ will be max at $x=6$ sec since $s'(t)>0 \rightarrow s'(t)<0$ or $v(t)>0 \rightarrow v(t)<0$

* Consider endpts

$s(0)=\int_0^0 f(x)dx=0$

$s(9)<s(6)$ since s decreases on $(6,9)$.

e. Approximately when is the acceleration zero?

$a(t)=0$ when $v'(t)=0$. Acceleration zero at approx $t=4,7$ sec.

f. when is the particle moving toward the origin? Away from the origin?

Since $s(0)=0$ and $s'(t)>0$ on $(0,6)$, the particle is moving away. On $(6,9)$, $s'(t)<0$ so the particle is moving toward the origin.

g. On which side of the origin does the particle lie at time $t=9$?

The particle is to the right of the origin at $t=9$.

$s(9)=\int_0^9 f(t)dt=\int_0^6 f(t)dt+\int_6^9 f(t)dt$. Since $\underbrace{\int_0^6 f(t)dt}_{\oplus \text{ area}} > \underbrace{\int_6^9 f(t)dt}_{\ominus \text{ area}}$ – net displ. is $\oplus$