Do Now: Find the solution to the following differential equations given an initial condition.

1) $\frac{dy}{dx} = -\frac{x}{y}$ and $y = -3$ when $x = 4$

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$

$y^2 = -x^2 + C$

$(-3)^2 = -(4)^2 + C$

$9 = -16 + C$

$25 = C$

$y^2 = -x^2 + 25$

$y = \pm\sqrt{25 - x^2}$

Since $(4, -3)$

$\boxed{y = -\sqrt{25 - x^2}}$

2) $\frac{dy}{dx} = (y+5)(x+2)$ and $y = 1$ when $x = 0$

$\int \frac{dy}{y+5} = \int (x+2)\,dx$

$\ln|y+5| = \frac{x^2}{2} + 2x + C_1$

$|y+5| = e^{\frac{x^2}{2} + 2x + C_1}$

$y = \pm e^{x^2/2 + 2x} e^{C_1} - 5$

$y = Ce^{x^2/2 + 2x} - 5$

$(0, 1)$

$1 = Ce^0 - 5$

$C = 6$

$\boxed{y = 6e^{x^2/2 + 2x} + 5}$

Class Work/Homework:

1) Find the solution of the differential equation $\frac{dy}{dt} = ky$, k a constant, that satisfies the given conditions: $k = 1.5$, $y(0) = 100$.

$\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + C_1$

$|y| = e^{kt + C_1}$

$y = \pm e^{1.5t} e^{C_1}$

$y = Ce^{1.5t}$

$100 = Ce^0$

$C = 100$

$\boxed{y(t) = 100e^{1.5t}}$

2) Solve the differential equation $y' + 2y = 0$. Find the particular solution that goes through $(-1, e^2)$.

$\frac{dy}{dx} + 2y = 0$

$\frac{dy}{dx} = -2y$

$\int \frac{dy}{y} = \int -2\,dx$

$\ln|y| = -2x + C_1$

$|y| = e^{-2x + C_1}$

$y = \pm e^{-2x} e^{C_1}$

$y = Ce^{-2x}$

$(-1, e^2)$

$e^2 = Ce^{-2(-1)}$

$1 = C$

$\boxed{y = e^{-2x}}$

3) Determine whether $y = 4e^{-x}$ is a solution of the differential equation $y'' - y = 0$.

$y' = -4e^{-x}$

$y'' = 4e^{-x}$

$y'' - y \overset{?}{=} 0$

$4e^{-x} - 4e^{-x} = 0$

$0 = 0$ ✓ yes!

For 4-9, solve the differential equations with given condition.

4) $\frac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$

$\int \frac{dy}{\cos^2 y} = \int dx$

$\int \sec^2 y \, dy = \int dx$

$\tan y = x + C$

$\tan(0) = C$

$C = 0$

$\boxed{y = \tan^{-1}(x)}$

5) $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$

$= (\cos x)e^{y}e^{\sin x}$

$u = \sin x$ $\quad dx = \frac{du}{\cos x}$

$\frac{du}{dx} = \cos x$

$\int \frac{dy}{e^y} = \int \cos x \, e^{\sin x} dx$

$\int e^{-y} dy = \int e^u du$

$-e^{-y} = e^u + C$

$-e^{-y} = e^{\sin x} + C$

$-1 = 1 + C$

$C = -2$

$-e^{-y} = e^{\sin x} - 2$

$e^{-y} = 2 - e^{\sin x}$

$-y = \ln(2 - e^{\sin x})$

$\boxed{y = -\ln(2 - e^{\sin x})}$

6) $\frac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$

$\frac{dy}{dx} = e^x e^{-y}$

$\int \frac{dy}{e^{-y}} = \int e^x dx$

$e^y = e^x + C$

$e^2 = 1 + C$

$C = e^2 - 1$

$e^y = e^x + e^2 - 1$

$\boxed{y = \ln(e^x + e^2 - 1)}$

7) $\frac{dy}{dx} = -2xy^2$ with initial condition $f(1) = \frac{1}{4}$

$\int \frac{dy}{y^2} = \int -2x \, dx$

$-\frac{1}{y} = -x^2 + C$

$-4 = -1 + C$

$C = -3$

$-\frac{1}{y} = -x^2 - 3$

$\frac{1}{y} = x^2 + 3$

$\boxed{y = \frac{1}{x^2+3}}$

8) $xy' - 3y = 0$ and $y = 2$ when $x = -3$

$\frac{dy}{dx} = \frac{3y}{x}$

$\int \frac{dy}{y} = \int \frac{3}{x} dx$

$\ln|y| = 3\ln|x| + C_1$

$\ln|y| = \ln|x^3| + C_1$

$|y| = e^{\ln|x^3| + C_1}$

$y = \pm e^{\ln|x^3|} e^{C_1}$

$y = C|x^3|$

$2 = C|(-3)^3|$

$\frac{2}{27} = C$

$\boxed{y = \frac{2}{27}|x^3|}$

9) $\frac{dy}{dx} = -\frac{2x}{y}$ with initial condition $f(1) = -1$

$\int y \, dy = \int -2x \, dx$

$\frac{y^2}{2} = -x^2 + C$

$\frac{1}{2} = -1 + C$

$C = \frac{3}{2}$

$\frac{y^2}{2} = -x^2 + \frac{3}{2}$

$y^2 = 3 - 2x^2$

$y = \pm\sqrt{3 - 2x^2}$

Given $(1, -1)$

$\boxed{y = -\sqrt{3 - 2x^2}}$

10) Let f be a function with $f(1)=4$ such that for all points (x,y) on the graph of f the slope is given by $\frac{3x^2+1}{2y}$.

a. Find the slope of the graph of f at the point where $x=1$.

$$\left.\frac{dy}{dx}\right|_{(1,4)} = \frac{3+1}{2(4)} = \boxed{\frac{1}{2}}$$

b. Write an equation for the line tangent to the graph of f at $x=1$ and use it to approximate $f(1.2)$.

$$y-4=\tfrac{1}{2}(x-1)$$
$$L(x)=\tfrac{1}{2}(x-1)+4$$

$$f(1.2)\approx L(1.2)=\tfrac{1}{2}\left(\tfrac{2}{10}\right)+4$$
$$\boxed{f(1.2)\approx 4.1}$$

c. Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx}=\frac{3x^2+1}{2y}$ with the initial condition $f(1)=4$.

$$\int 2y\,dy=\int(3x^2+1)\,dx$$
$$y^2=x^3+x+C$$
$$16=1+1+C$$
$$C=14$$

$$y^2=x^3+x+14$$
$$y=\pm\sqrt{x^3+x+14}$$
given $(1,4)$

$$y=\sqrt{x^3+x+14}$$
$$\boxed{f(x)=\sqrt{x^3+x+14}}$$

d. Use your solution from part (c) to find $f(1.2)$. What can you conclude about the concavity of the function?

$$f(1.2)\approx 4.114$$

Since $f(1.2)>L(1.2)$, the tangent line is below the curve and the function is concave up.

e. Confirm your conclusion regarding the concavity by finding $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{3x^2+1}{2y}\right)$$

$$\frac{d^2y}{dx^2}=\frac{6x(2y)+(3x^2+1)\left(2\frac{dy}{dx}\right)}{(2y)^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(12xy+2(3x^2+1)\left(\frac{3x^2+1}{2y}\right)\right)2y}{(4y^2)2y}$$

$$\frac{d^2y}{dx^2}=\frac{\cancel{24}^{12}xy^2+\cancel{2}^{1}(9x^4+6x^2+1)}{4\cancel{8}y^3}$$

$$\frac{d^2y}{dx^2}=\frac{12xy^2+9x^4+6x^2+1}{4y^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,4)}=\frac{192+9+6+1}{256}>0$$ Concave up!

(a) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ $\left.\dfrac{dy}{dx}\right\rvert_{\substack{x=1\\y=4}} = \dfrac{3+1}{2\cdot 4} = \dfrac{4}{8} = \dfrac{1}{2}$	**1**: answer
(b) $y - 4 = \dfrac{1}{2}(x-1)$ $f(1.2) - 4 \approx \dfrac{1}{2}(1.2-1)$ $f(1.2) \approx 0.1 + 4 = 4.1$	$\mathbf{2}\begin{cases} 1\text{: equation of tangent line} \\ 1\text{: uses equation to approximate } f(1.2) \end{cases}$
(c) $2y\,dy = (3x^2+1)\,dx$ $\int 2y\,dy = \int (3x^2+1)\,dx$ $y^2 = x^3 + x + C$ $4^2 = 1 + 1 + C$ $14 = C$ $y^2 = x^3 + x + 14$ $y = \sqrt{x^3+x+14}$ is branch with point $(1,4)$ $f(x) = \sqrt{x^3+x+14}$	$\mathbf{5}\begin{cases} 1\text{: separates variables} \\ 1\text{: antiderivative of } dy \text{ term} \\ 1\text{: antiderivative of } dx \text{ term} \\ 1\text{: uses } y=4 \text{ when } x=1 \text{ to pick one} \\ \quad\ \text{function out of a family of functions} \\ 1\text{: solves for } y \\ \quad\ 0/1 \text{ if solving a linear equation in } y \\ \quad\ 0/1 \text{ if no constant of integration} \end{cases}$ Note: max 0/5 if no separation of variables Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation
(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$	**1**: answer, from student's solution to the given differential equation in (c)