Find the solution to the following differential equations given an initial condition.

1) $\frac{dy}{dx} = 12x + 3$; $f(-1) = 8$

$\int dy = \int (12x+3)\,dx$

$y = 6x^2 + 3x + C$

$8 = 6(-1)^2 + 3(-1) + C$

$5 = C$

$\boxed{y = 6x^2 + 3x + 5}$

2) $\frac{dy}{dx} = \cos x + 3$; $f\left(\frac{\pi}{6}\right) = \frac{5+\pi}{2}$

$\int dy = \int (\cos x + 3)\,dx$

$y = \sin x + 3x + C$

$\frac{5+\pi}{2} = \sin \pi/6 + \frac{3\pi}{6} + C$

$\frac{5+\pi}{2} = \frac{1+\pi}{2} + C$

$5+\pi = 1 + \pi + 2C$

$4 = 2C \longrightarrow C = 2$

$\boxed{y = \sin x + 3x + 2}$

Check:

$\frac{5+\pi}{2} = \sin\frac{\pi}{6} + \frac{\pi}{2} + 2$

$\frac{5+\pi}{2} = \frac{1}{2} + \frac{\pi}{2} + \frac{4}{2}$

$\frac{5+\pi}{2} = \frac{5+\pi}{2}$ ✓

Notes:

> A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ is called **separable**. We separate the variables by writing it in the form:
>
> $$\frac{1}{f(y)}dy = g(x)dx$$
>
> The solution is found by integrating each side with respect to its isolated variable.

Example 1: Solve for y if $\frac{dy}{dx} = (xy)^2$ and $y = 1$ when $x = 1$. $(1,1)$

$\frac{dy}{dx} = x^2y^2$

$\int \frac{dy}{y^2} = \int x^2\,dx$

$-\frac{1}{y} = \frac{x^3}{3} + C$

$-1 = \frac{1}{3} + C$

$-\frac{4}{3} = C$

$-\frac{1}{y} = \frac{x^3}{3} - \frac{4}{3}$

$-y = \frac{3}{x^3 - 4}$

$\boxed{y = \frac{3}{4 - x^3}}$

Example 2: Use separation of variables to solve $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$. $(1,2)$

$\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$\frac{2^2}{2} = \frac{1}{2} + C$

$\frac{3}{2} = C$

$\frac{y^2}{2} = \frac{x^2}{2} + \frac{3}{2}$

$y^2 = x^2 + 3$

$y = \pm\sqrt{x^2+3}$

Using given condition $(1,2)$

$\boxed{y = \sqrt{x^2+3}}$