Answer Key

Do Now:

1. The rate of change of the amount of a substance W is directly proportional to the amount of the substance present at a given time t, meaning $\frac{dW}{dt} = kW$. Find the equation that models the amount of substance present at time t.

$\int \frac{dW}{W} = \int k\, dt$

$\ln|W| = kt + C_1$

$|W| = e^{kt+C_1}$

$W = \pm e^{kt} e^{C_1}$

$\boxed{W = Ce^{kt}}$

2. Based on the equation found in #1, when would the model represent a growth function? When would it represent a decay function?

Growth function when $k>0$

Decay function when $k<0$

3. Find the particular solution of the equation in #1 if there are 250 gallons of the substance present when $t = 0$.

$(0, 250)$

$250 = Ce^0$

$C = 250$

$\boxed{W = 250e^{kt}}$

In general, $y = y_0 e^{kt}$ is exponential growth/decay model.

Class Work:

1. Suppose that the cholera bacteria in a colony grows unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hours?

Let B = # bacteria, t = time

$B = Ce^{kt}$

Given: $B(0) = 1$

$B(\frac{1}{2}) = 2$

Find: $B(24)$

$\underline{B(0) = 1}$

$1 = Ce^0$

$B = e^{kt}$

$\underline{B(\frac{1}{2}) = 2}$

$2 = e^{\frac{1}{2}k}$

$\ln 2 = \frac{1}{2}k$

$k = 2\ln 2 = \ln 4$

$B = e^{t\ln 4}$

$B = e^{\ln 4^t}$

$B = 4^t$

$B(24) = 4^{24}$

$\boxed{B(24) = 2.8 \times 10^{14} \text{ bacteria}}$

2. The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the amount of radon in a sealed sample of air to decay to 90% of its original value?

Find t when $y = .9y_0$

$.9y_0 = y_0 e^{-.18t}$

$\ln(.9) = -.18t$

$t = \frac{\ln(.9)}{-.18}$

$\boxed{t \approx .585 \text{ days}}$

3. The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. Given, the half-life of carbon-14 is 5,700 years, about how old is Crater Lake?

$y = y_0 e^{kt}$

Given: $y(5700) = \frac{1}{2} y_0$

Find: t when $y = .445 y_0$

$y(5700) = \frac{1}{2} y_0$

$\frac{1}{2}\cancel{y_0} = \cancel{y_0} e^{5700k}$

$\ln(\frac{1}{2}) = 5700k$

$k = \frac{\ln(\frac{1}{2})}{5700}$

$y = y_0 e^{\frac{t}{5700}\ln(\frac{1}{2})}$

$.445\cancel{y_0} = \cancel{y_0} e^{\frac{t}{5700}\ln(\frac{1}{2})}$

$\ln .445 = \frac{t}{5700}\ln(\frac{1}{2})$

$t = 5700 \frac{\ln(.445)}{\ln(\frac{1}{2})}$

$\boxed{t \approx 6658.300 \text{ yrs}}$

4. The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

$\frac{dN}{dt} = k(650 - N)$

$\int \frac{dN}{650 - N} = \int k\,dt$

$-\ln|650 - N| = kt + C_1$

$\ln|650 - N| = -kt + C_2$

$|650 - N| = e^{-kt + C_2}$

$650 - N = \pm e^{-kt} e^{C_2}$

$N = 650 - Ce^{-kt}$

$(0, 300)$

$300 = 650 - Ce^0$

$C = 350$

$N = 650 - 350e^{-kt}$

$(2, 500)$

$500 = 650 - 350e^{-2k}$

$\frac{3}{7} = e^{-2k} \rightarrow \ln(\frac{3}{7}) = -2k$

$-k = \ln(\frac{3}{7})^{\frac{1}{2}}$

$N(t) = 650 - 350 e^{t\ln(\frac{3}{7})^{\frac{1}{2}}}$

$\boxed{N(t) = 650 - 350(\frac{3}{7})^{t/2}}$

$N(3) = 552$ Coyotes

5. Ignoring resistance, a sailboat starting from rest (meaning its initial velocity is zero) accelerates, $\frac{dV}{dt}$, at a rate proportional to the difference between the velocities of the wind (W) and the boat (V). The wind is blowing at a constant 20 knots, and after 1 minute the boat is moving at 5 knots.

a. Write the velocity v as a function of time t.

b. Write the distance traveled by the boat as a function of time.

a) $\frac{dV}{dt} = k(W - V)$

$\frac{dV}{W - V} = k\,dt$

$-\ln|W - V| = kt + C_1$

$\ln|W - V| = -kt + C_2$

$|W - V| = e^{-kt + C_2}$

$W - V = \pm e^{-kt} e^{C_2}$

$W - V = Ce^{-kt}$

$\boxed{V = W - Ce^{-kt}}$

Since $W = 20$ knots

$V = 20 - Ce^{-kt}$

$(0,0)$ initial vel $= 0$

$0 = 20 - Ce^0$

$C = 20$

$V = 20 - 20e^{-kt}$

$(1, 5)$

$5 = 20 - 20e^{-k}$

$\frac{3}{4} = e^{-k} \Rightarrow \ln(\frac{3}{4}) = -k$

$V = 20 - 20e^{t\ln(3/4)}$

$\boxed{V = 20 - 20(\frac{3}{4})^t}$

b) $\int V\,dt = \int 20 - 20(\frac{3}{4})^t$

$D = 20(t - \frac{(\frac{3}{4})^t}{\ln(3/4)}) + C$

$(0,0)$ $0 = 20(-\frac{1}{\ln(3/4)}) + C$

$C = \frac{20}{\ln(3/4)}$

$\boxed{D = 20t - \frac{20(\frac{3}{4})^t}{\ln(3/4)} + \frac{20}{\ln(3/4)}}$

6. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2(v+17)$, with initial condition $v(0) = -47$.

   a. Find an expression v in terms of t, where t is measured in seconds.

   b. Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

   c. It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

a) $\frac{dV}{dt} = -2(V+17)$

$\int \frac{dV}{V+17} = \int -2\,dt$

$\ln|V+17| = -2t + C_1$

$|V+17| = e^{-2t+C_1}$

$V+17 = \pm e^{-2t} e^{C_1}$

$V = Ce^{-2t} - 17$

$(0,-47) \Rightarrow \quad C = -30$

$\boxed{V = -30e^{-2t} - 17}$

b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(\frac{-30}{e^{2t}} - 17\right) = -17 \text{ ft/s}$

c) $^{*}-20 = -30e^{-2t} - 17$

* Direction! Heading toward ground.

$-3 = -30e^{-2t}$

$\frac{1}{10} = e^{-2t}$

$\ln\left(\frac{1}{10}\right) = -2t$

$t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$

$\boxed{t \approx 1.151 \text{ sec}}$

7. A population P of wolves at time t years $(t \geq 0)$ is increasing a rate directly proportional to $600 - P$, where the constant of proportionality is k.

   a. If $P(0) = 200$, find $P(t)$ in terms of t and k.

   b. If $P(2) = 500$, find k.

   c. Find $\lim_{t\to\infty} P(t)$

a) $\frac{dP}{dt} = k(600 - P)$

$\int \frac{dP}{600-P} = \int k\,dt$

$-\ln|600-P| = kt + C_1$

$|600-P| = e^{-kt+C_1}$

$600 - P = \pm e^{-kt} e^{C_1}$

$P = 600 - Ce^{-kt}$

$(0,200) \quad 200 = 600 - Ce^{0}$

$C = 400 \Rightarrow \quad \boxed{P = 600 - 400e^{-kt}}$

b) $(2, 500)$

$500 = 600 - 400e^{-2k}$

$\frac{1}{4} = e^{-2k}$

$\ln\left(\frac{1}{4}\right) = -2k$

$-k = \frac{1}{2}\ln\left(\frac{1}{4}\right) = \ln\left(\frac{1}{4}\right)^{\frac{1}{2}} = \ln\left(\frac{1}{2}\right)$

$P = 600 - 400e^{t\ln\left(\frac{1}{2}\right)} = 600 - 400e^{\ln\left(\frac{1}{2}\right)^t}$

$\boxed{P = 600 - 400\left(\frac{1}{2}\right)^t}$

c) $\lim_{t\to\infty}\left(600 - \underbrace{400\left(\frac{1}{2}\right)^t}_{\text{approches zero}}\right)$

$= \boxed{600 \text{ wolves}}$

8. An automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 mph, the number of mpg drops at the rate of 12% for each 10 mph.

a. s is the speed and y is the number of mpg. Find y as a function of s if $\frac{dy}{ds} = -.012y, \quad s > 50$.

b. Use the function in part (a) to complete the table.

Speed	50	55	60	65	70
Miles per Gallon	28	26.369	24.834	23.388	22.026

a) $\int \frac{dy}{y} = \int -.012\,ds$

$\ln|y| = -.012s + C_1$

$|y| = e^{-.012s + C_1}$

$y = \pm e^{-.012s} e^{C_1}$

$y = Ce^{-.012s}$

$(50, 28)$

$y(50) = 28$

$28 = Ce^{-.012(50)}$

$C = 28e^{.6}$

$\boxed{y = 28e^{.6 - .012s}}$