Do Now:

A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000 bacteria. How many bacteria were initially present?

$(3, 10{,}000)$
$(5, 40{,}000)$

$y = y_0 e^{kt}$

$10{,}000 = y_0 e^{k(3)}$ $\qquad$ $40{,}000 = y_0 e^{k(5)}$

$y_0 = \frac{10{,}000}{e^{3k}}$ $\qquad = \qquad$ $y_0 = \frac{40{,}000}{e^{5k}}$

$\frac{10{,}000}{e^{3k}} = \frac{40{,}000}{e^{5k}}$

$\frac{e^{5k}}{e^{3k}} = \frac{40{,}000}{10{,}000}$

$e^{2k} = 4$

$2k = \ln 4$
$k = \frac{1}{2}\ln(4)$
$k = \ln(2)$

$10{,}000 = y_0 e^{3\ln 2}$
$10{,}000 = y_0 e^{\ln 2^3}$
$10{,}000 = y_0 (8)$
$y_0 = 1250$ bacteria

Notes:

The rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is **Newton's Law of Cooling**, although it applies to warming as well.

Let T be the temperature of the object at time t, with T_S denoting the surrounding temperature. If T_0 is the initial temperature at $t = 0$, then Newton's Law of Cooling can be represented by the equation:

$$T - T_S = (T_0 - T_S)e^{-kt}$$

Examples:

1. Mrs. Canonaco loves her oatmeal in the morning! Her kitchen is kept at a constant 60°F. If her bowl of oatmeal cools from 100°F to 90°F in 10 minutes, how much longer will it take for its temperature to decrease to 80°F so she can enjoy it before school?

$T_s = 60°F$
$T_0 = 100°F$
$T(10 \text{min}) = 90°F$

$90 - 60 = (100 - 60)e^{-10k}$

$\frac{3}{4} = e^{-10k}$

$-10k = \ln(\frac{3}{4})$

$-k = \frac{1}{10}\ln(\frac{3}{4})$

$T - 60° = (T_0 - 60°)e^{\frac{t}{10}\ln(\frac{3}{4})}$

$T = 80°, T_s = 90°$

$80 - 60 = (90 - 60)e^{\frac{t}{10}\ln(\frac{3}{4})}$

$\frac{2}{3} = e^{\frac{t}{10}\ln(\frac{3}{4})}$

$\ln(\frac{2}{3}) = \frac{t}{10}\ln(\frac{3}{4})$

$t = \frac{10\ln(\frac{2}{3})}{\ln(\frac{3}{4})} \approx 14.094$

$\boxed{\text{It will take an additional 14 mins.}}$

2. Mr. Ursino enjoys his hot cocoa! He sips his cocoa in a Starbucks which is kept at a constant 70°F. It takes 10 minutes for the hot cocoa to cool from 120°F to 110°F. How many more minutes will it take to cool to 100°F?

$T_s = 70°$

$T_0 = 120°$

$T(10\text{min}) = 110°F$

$110 - 70 = (120-70)e^{-10k}$

$\frac{4}{5} = e^{-10k}$

$\ln\left(\frac{4}{5}\right) = -10k$

$-k = \frac{1}{10}\ln\left(\frac{4}{5}\right)$

$T - 70° = (T_0 - 70)e^{\frac{t}{10}\ln\left(\frac{4}{5}\right)}$

$T = 100°F,\ T_0 = 110°F$

$100 - 70 = (110-70)e^{\frac{t}{10}\ln\left(\frac{4}{5}\right)}$

$\frac{3}{4} = e^{\frac{t}{10}\ln\left(\frac{4}{5}\right)}$

$\ln\left(\frac{3}{4}\right) = \frac{t}{10}\ln\left(\frac{4}{5}\right)$

$t = 10\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{4}{5}\right)} \approx 12.892$

$\boxed{\text{It will take another 12.9 min to cool.}}$

3. Renesh and JP carry an aluminum beam from the cold outside into a machine shop where the temperature was held at 65°F. After 10 minutes, the beam warmed to 35°F and after another 10 minutes its temperature was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature.

$T_s = 65°F$

$T(10\text{min}) = 35°F$

$T_s = 65°F$

$T(20\text{min}) = 50°F$

Find T_0

$35 - 65 = (T_0 - 65)e^{-10k}$

$\frac{-30}{e^{-10k}} = T_0 - 65$

$50 - 65 = (T_0 - 65)e^{-20k}$

$\frac{-15}{e^{-20k}} = T_0 - 65$

$\frac{-30}{e^{-10k}} = \frac{-15}{e^{-20k}}$

$\frac{30}{15} = \frac{e^{-10k}}{e^{-20k}}$

$2 = e^{10k}$

$\ln(2) = 10k$

$k = \frac{1}{10}\ln(2)$

$\frac{-30}{e^{\ln(2)}} = T_0 - 65$

$\frac{-30}{\frac{1}{2}} = T_0 - 65$

$-60 + 65 = T_0$

$\boxed{T_0 = 5°F}$

$$T - T_s = (T_0 - T_s)e^{-kt}$$

a)
4. Suppose that Ashika's cup of soup cooled from 90°C to 60°C in 10 minutes in her kitchen where the temperature was 20°C. Use Newton's Law of Cooling to figure out how much longer it would take the soup to cool to 35°C. b) Instead of being left to stand in the kitchen, Ashika puts the cup of 90°C soup into a freezer with a temperature of -15°C. How long will it take the soup to cool from 90°C to 35°C?

a) $T_s = 20°C$
$T_0 = 90°C$
$T(10\,min) = 60°C$

$60 - 20 = (90 - 20)e^{-10k}$
$\frac{4}{7} = e^{-10k}$
$\ln(\frac{4}{7}) = -10k$
$-k = \frac{1}{10}\ln(\frac{4}{7})$

$t = ?$ $T = 35°C$
$T_0 = 60°C$
$T - 20 = (T_0 - 20)e^{\frac{t}{10}\ln(\frac{4}{7})}$
$35 - 20 = 60 - 20\ e^{\frac{t}{10}\ln(\frac{4}{7})}$
$\frac{15}{40} = \frac{3}{8} = e^{\frac{t}{10}\ln(\frac{4}{7})}$
$\ln(\frac{3}{8}) = \frac{t}{10}\ln(\frac{4}{7})$
$t = \frac{10\ln(\frac{3}{8})}{\ln(\frac{4}{7})} \approx 17.527$ min

$\boxed{\text{It will take an additional 17.5 min}}$

b) $T_s = -15°C$ $t = ?$ $T = 35°C$
$T_0 = 90°C$

$35 + 15 = (90 + 15)e^{\frac{t}{10}\ln(\frac{4}{7})}$
$\frac{50}{105} = \frac{10}{21} = e^{\frac{t}{10}\ln(\frac{4}{7})}$

$\ln(\frac{10}{21}) = \frac{t}{10}\ln(\frac{4}{7})$
$t = \frac{10\ln(\frac{10}{21})}{\ln(\frac{4}{7})}$
$\boxed{t \approx 13.258\,min}$

5. The temperature of Sarika's ingot of silver is 60°C above room temperature right now. Twenty minutes ago it was 70°C above room temperature. How far above room temperature will the silver be 15 minutes from now? 2 hours from now? When will the silver be 10°C above room temperature?

$t = 20\,min$ $T = 60° + T_s$
$t = 0\,min$ $T_0 = 70° + T_s$
$T(15) = ?$

$60 + T_s - T_s = (70 + T_s - T_s)e^{-20k}$
$\frac{6}{7} = e^{-20k}$
$\ln(\frac{6}{7}) = -20k$
$-k = \frac{1}{20}\ln(\frac{6}{7})$

$\boxed{T - T_s = (70)e^{\frac{t}{20}\ln(\frac{6}{7})}}$

$t = 35\,min$
$T - T_s = 70\left(\frac{6}{7}\right)^{\frac{35}{20}}$
$\boxed{\approx 53.449°C \text{ above room temp}}$

$t = 140\,min$
$T - T_s = 70\left(\frac{6}{7}\right)^{\frac{14}{2}}$
$\boxed{\approx 23.794°C \text{ above room temp}}$

$t = ?$ $T - T_s = 10°C$
$10 = 70\left(\frac{6}{7}\right)^{t/20}$
$\frac{1}{7} = \frac{6}{7}^{\frac{t}{20}}$
$\frac{20\ln(\frac{1}{7})}{\ln(\frac{6}{7})} = t$
$\boxed{t \approx 232.469\,min \approx 3.874\,hrs}$