<u>Do Now</u>:

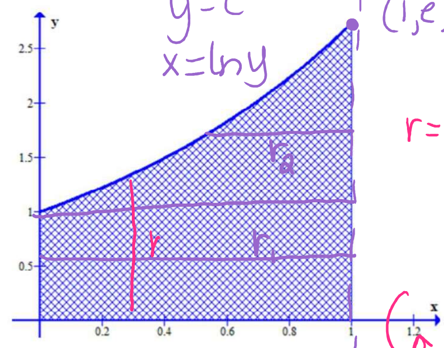
1) Set-up, but do not solve, an integral that will find the volume of the solid obtained by rotating the region formed between $f(x)=e^x$ and the x-axis about the given axis over the interval $[0,1]$.

   a. x-axis

   $V=\pi\int_0^1 (e^x)^2\,dx=\pi\int_0^1 e^{2x}\,dx$

   b. $x=1$

   $V=\pi\int_0^1 1^2\,dy+\pi\int_1^e (1-\ln y)^2\,dy$

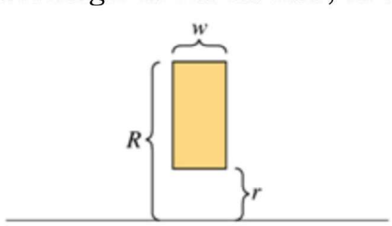


$y=e^x$
$x=\ln y$
$(1,e)$
$r=e^x$
$r_1=1$
$r_2=1-x=1-\ln y$

2) A solid object is formed by taking out a cylinder of height 7 inches and radius 2 inches from the inside of a solid cylinder with a height of 7 inches and a radius of 5 inches. Find the volume of the remaining solid.

<u>Notes</u>:

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer.** The washer is formed by revolving a rectangle about an axis, as shown in the figure below.

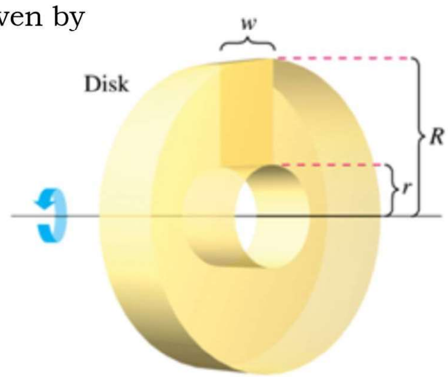


Axis of revolution

If r and R are the inner and outer radii of the washer and w is the width of the washer, the volume is given by:

<u>Volume of Washer</u> = $\pi\left(R^2-r^2\right)w$

To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$, as shown in the figure below. If the figure is revolved about its axis of revolution, the volume of the resulting solid is given by

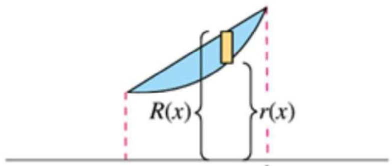


Solid of revolution

<u>Washer Method</u>

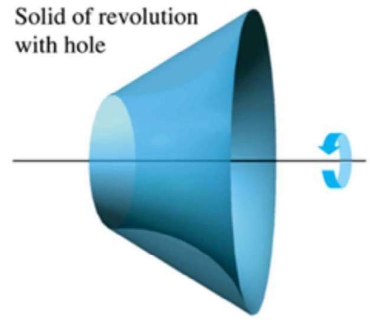
$$\pi\int_a^b \left(R^2(x)-r^2(x)\right)dx$$

Note that the integral involving the inner radius represents the volume of the hole and is *subtracted* from the integral involving the outer radius.



Plane region

Solid of revolution with hole

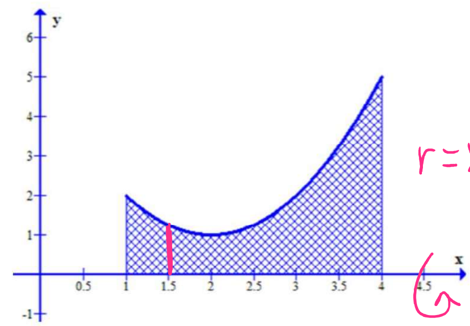


Example 1: Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, $y = 0$ about:

a) the x-axis

$$V = \pi \int_1^4 (x^2 - 4x + 5)^2 \, dx$$

$$V = \boxed{\frac{78\pi}{5}}$$



$r = x^2 - 4x + 5$

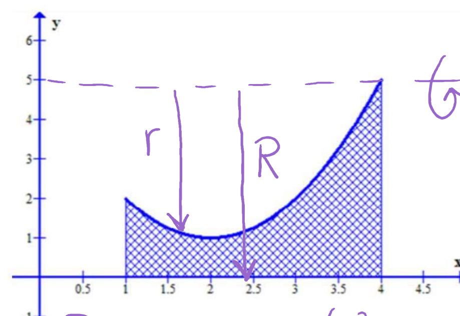
b) $y = 5$

$$V = \pi \int_1^4 5^2 - (-x^2 + 4x)^2 \, dx$$

$x^4 - 8x^3 + 16x^2$

$$V = \pi \int_1^4 (25 - x^4 + 8x^3 - 16x^2) \, dx$$

$$V = \boxed{44.4\pi = \frac{222\pi}{5}}$$



$R = 5 \qquad r = 5 - (x^2 - 4x + 5)$

$r = -x^2 + 4x$

c) $y = -1$

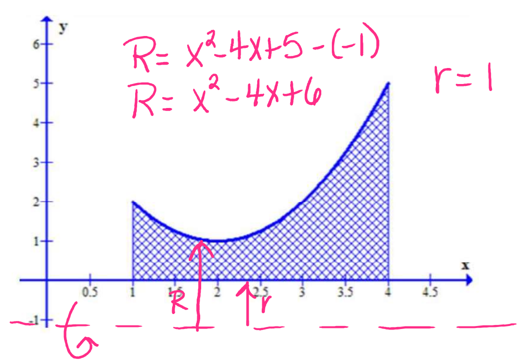
$$V = \pi \int_1^4 \left[(x^2 - 4x + 6)^2 - 1^2\right] dx$$

$$V = \boxed{27.6\pi = \frac{138\pi}{5}}$$

$R = x^2 - 4x + 5 - (-1)$

$R = x^2 - 4x + 6$

$r = 1$



Example 2: Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$ that lies in the first quadrant about:

a) the x-axis

$\sqrt[3]{x} = \dfrac{x}{4}$

$4x^{1/3} = x$

$0 = x - 4x^{1/3}$

$0 = x^{1/3}\left(x^{2/3} - 4\right)$

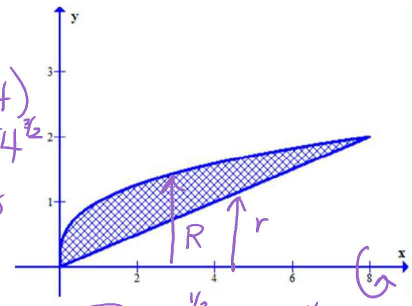
$x = 0 \quad \left(x^{2/3}\right)^{3/2} = 4^{3/2}$

$x = 8$

$$V = \pi \int_0^8 \left(x^{1/3}\right)^2 - \left(\frac{x}{4}\right)^2 dx$$

$$V = \pi \int_0^8 \left(x^{2/3} - \frac{x^2}{16}\right) dx$$

$$V = \pi \left(\frac{3}{5}x^{5/3} - \frac{x^3}{48}\right)\Bigg|_0^8 = \boxed{\frac{128\pi}{15}}$$



$R = x^{1/3} \quad r = \dfrac{x}{4}$

b) the y-axis

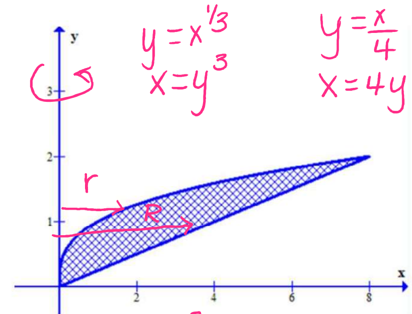
$y = x^{1/3} \quad y = \dfrac{x}{4}$

$x = y^3 \quad x = 4y$

$$V = \pi \int_0^2 (4y)^2 - \left(y^3\right)^2 dy$$

$$V = \pi \int_0^2 \left(16y^2 - y^6\right) dy$$

$$V = \pi \left(\frac{16}{3}y^3 - \frac{y^7}{7}\right)\Bigg|_0^2 = \boxed{\frac{512\pi}{21}}$$



$r = y^3 \quad R = 4y$

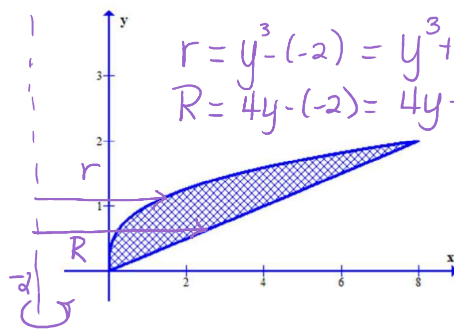
c) $x = -2$

$r = y^3 - (-2) = y^3 + 2$

$R = 4y - (-2) = 4y + 2$

$$V = \pi \int_0^2 (4y+2)^2 - \left(y^3+2\right)^2 dy$$

$$V = \boxed{\frac{848\pi}{21}}$$



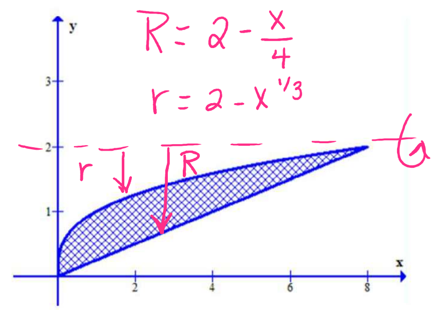
d) $y = 2$

$R = 2 - \dfrac{x}{4}$

$r = 2 - x^{1/3}$

$$V = \pi \int_0^8 \left(2 - \frac{x}{4}\right)^2 - \left(2 - x^{1/3}\right)^2 dx$$

$$V = \boxed{\frac{112\pi}{15}}$$



Example 3: Determine the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 2x$ about:

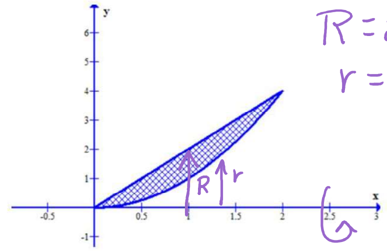
1) The x-axis

$$V = \pi \int_0^2 (2x)^2 - (x^2)^2 \, dx$$

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0, 2$$

$$V = \pi \int_0^2 (4x^2 - x^4)\, dx$$

$$\boxed{V = \frac{64\pi}{15}}$$



$R = 2x$
$r = x^2$

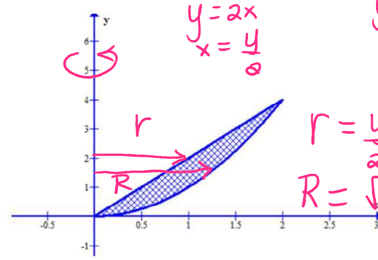
2) The y-axis

$$V = \pi \int_0^4 (\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 dy = \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy$$

$$V = \boxed{\frac{8\pi}{3}}$$

$y = 2x$
$x = \frac{y}{2}$

$y = x^2$
$x = \sqrt{y}$

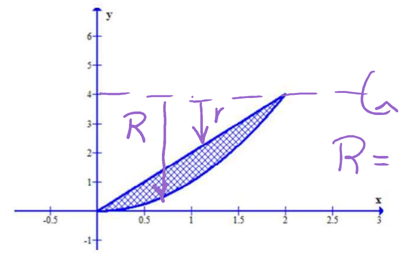


$r = \frac{y}{2}$
$R = \sqrt{y}$

3) $y = 4$

$$V = \pi \int_0^2 (4 - x^2)^2 - (4 - 2x)^2 \, dx$$

$$V = \boxed{\frac{32\pi}{5}}$$

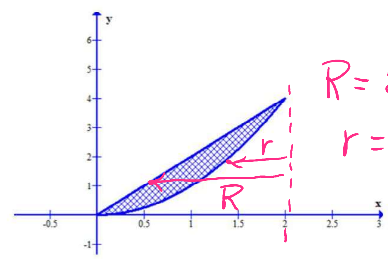


$R = 4 - x^2$
$r = 4 - 2x$

4) $x = 2$

$$V = \pi \int_0^4 (2 - y/2)^2 - (2 - \sqrt{y})^2 \, dy$$

$$V = \boxed{\frac{8\pi}{3}}$$

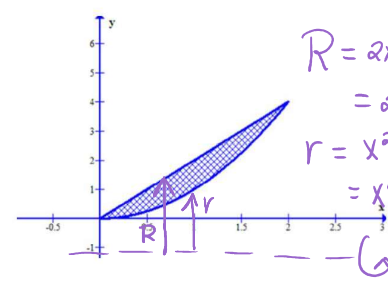


$R = 2 - y/2$
$r = 2 - \sqrt{y}$

5) $y = -1$

$$V = \pi \int_0^2 (2x+1)^2 - (x^2+1)^2 \, dx$$

$$V = \boxed{\frac{104\pi}{15}}$$

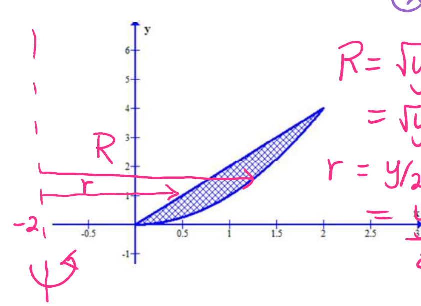


$R = 2x - (-1)$
$= 2x + 1$
$r = x^2 - (-1)$
$= x^2 + 1$

6) $x = -2$

$$V = \pi \int_0^4 (\sqrt{y} + 2)^2 - (y/2 + 2)^2 \, dy$$

$$V = \boxed{8\pi}$$



$R = \sqrt{y} - (-2)$
$= \sqrt{y} + 2$
$r = y/2 - (-2)$
$= \frac{y}{2} + 2$