AP Review #3 Answer Key with Scoring Rubrics

1. Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

(a) $\lim_{x \to 0^-} (1 - 2\sin x) = 1$

$\lim_{x \to 0^+} e^{-4x} = 1$

$f(0) = 1$

So, $\lim_{x \to 0} f(x) = f(0)$.

Therefore f is continuous at $x = 0$.

2 : analysis

(b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$-2\cos x \ne -3$ for all values of $x < 0$.

$-4e^{-4x} = -3$ when $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0$.

Therefore $f'(x) = -3$ for $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)$.

$3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$

(c)
$$\begin{aligned} \int_{-1}^{1} f(x)\,dx &= \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx \\ &= \int_{-1}^{0} (1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx \\ &= \Big[x + 2\cos x\Big]_{x=-1}^{x=0} + \Big[-\frac{1}{4}e^{-4x}\Big]_{x=0}^{x=1} \\ &= (3 - 2\cos(-1)) + \left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right) \end{aligned}$$

$$\begin{aligned} \text{Average value } &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx \\ &= \frac{13}{8} - \cos(-1) - \frac{1}{8}e^{-4} \end{aligned}$$

$4 : \begin{cases} 1 : \int_{-1}^{0} (1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

2. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

Solution	Scoring
(a) $\left.\frac{dW}{dt}\right\|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$ The tangent line is $y = 1400 + 44t$. $W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons	2 : { 1 : $\frac{dW}{dt}$ at $t = 0$ 1 : answer
(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$ Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$. The answer in part (a) is an underestimate.	2 : { 1 : $\frac{d^2W}{dt^2}$ 1 : answer with reason
(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ $\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$ $\ln\|W - 300\| = \frac{1}{25}t + C$ $\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$ $W - 300 = 1100e^{\frac{1}{25}t}$ $W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$	5 : { 1 : separation of variables 1 : antiderivatives 1 : constant of integration 1 : uses initial condition 1 : solves for W Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

3. a. $v'(t)=a(t)=-2e^{1-2t}$, $a(2)=-2e^{-3}$

   b. Since the acceleration and velocity are both negative, the speed is increasing.

   c.

$$v(t)=e^{1-2t}-\frac{1}{e}=0$$

$$e^{1-2t}=\frac{1}{e}$$

$$1-2t=\ln\left(\frac{1}{e}\right)$$

$$t=\frac{\ln\left(\frac{1}{e}\right)-1}{-2}=1$$

   Since $v(t)>0$ before $t=1$ and $v(t)<0$ after, the particle changes direction at $t=1$.

   d. $s(2)=1+\int_0^2\left(e^{1-2t}-\frac{1}{e}\right)dt=1.598$ units right

4.

(4) a) $\lim_{x\to 0^-} f(g(x))=\lim_{x\to 0^-} f(g(0^-))=f(1)=-1$
$\lim_{x\to 0^+} f(g(x))=\lim_{x\to 0^+} f(g(0^+))=f(2)=-1$
} $\lim_{x\to 0} f(g(x))=-1$

$f(g(0))=f(1)=-1$

Since $\lim_{x\to 0} f(g(x))=f(g(0))$, $f(g(x))$ is continuous at $x=0$

b) $\lim_{x\to 0^-} g(f(x))=\lim_{x\to 0^-} g(f(0^-))=g(0)=2$
$\lim_{x\to 0^+} g(f(x))=\lim_{x\to 0^+} g(f(0^+))=g(-1)=0$
} $\lim_{x\to 0} g(f(x))$ DNE

$g(f(x))$ is not continuous

c) $\lim_{x\to\infty} f(g(x))=-1$

$\lim_{x\to\infty} g(x)=\infty$ $\quad$ $\lim_{x\to\infty} f(x)=-1$

d) $\lim_{x\to 0^-} f(x)+g(x)=0+1=1$

$\lim_{x\to 0^+} k\,f(x)g(x)=k(-1)(2)=-2k$ $\quad$ $-2k=1$, $k=-\frac{1}{2}$

5. Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

Solution	Scoring
(a) $f'(1) = \left.\frac{dy}{dx}\right\|_{(1,\,2)} = 8$ An equation of the tangent line is $y = 2 + 8(x - 1)$.	$2: \begin{cases} 1: f'(1) \\ 1: \text{answer} \end{cases}$
(b) $f(1.1) \approx 2.8$ Since $y = f(x) > 0$ on the interval $1 \le x < 1.1$, $\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2) > 0$ on this interval. Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.	$2: \begin{cases} 1: \text{approximation} \\ 1: \text{conclusion with explanation} \end{cases}$
(c) $\frac{dy}{dx} = xy^3$ $\int \frac{1}{y^3}\,dy = \int x\,dx$ $-\frac{1}{2y^2} = \frac{x^2}{2} + C$ $-\frac{1}{2 \cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$ $y^2 = \frac{1}{\frac{5}{4} - x^2}$ $f(x) = \frac{2}{\sqrt{5 - 4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$	$5: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$ Note: max 2/5 [1-1-0-0-0] if no constant of integration Note: 0/5 if no separation of variables

6. The function g is defined for $x > 0$ with $g(1) = 2$, $g'(x) = \sin\left(x + \frac{1}{x}\right)$, and $g''(x) = \left(1 - \frac{1}{x^2}\right)\cos\left(x + \frac{1}{x}\right)$.

(a) Find all values of x in the interval $0.12 \le x \le 1$ at which the graph of g has a horizontal tangent line.

(b) On what subintervals of $(0.12, 1)$, if any, is the graph of g concave down? Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 0.3$.

(d) Does the line tangent to the graph of g at $x = 0.3$ lie above or below the graph of g for $0.3 < x < 1$? Why?

Solution	Scoring
(a) The graph of g has a horizontal tangent line when $g'(x) = 0$. This occurs at $x = 0.163$ and $x = 0.359$.	$2 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{answer} \end{cases}$
(b) $g''(x) = 0$ at $x = 0.129458$ and $x = 0.222734$ The graph of g is concave down on $(0.1295, 0.2227)$ because $g''(x) < 0$ on this interval.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$
(c) $g'(0.3) = -0.472161$ $g(0.3) = 2 + \int_1^{0.3} g'(x)\,dx = 1.546007$ An equation for the line tangent to the graph of g is $y = 1.546 - 0.472(x - 0.3)$.	$4 : \begin{cases} 1 : g'(0.3) \\ 1 : \text{integral expression} \\ 1 : g(0.3) \\ 1 : \text{equation} \end{cases}$
(d) $g''(x) > 0$ for $0.3 < x < 1$ Therefore the line tangent to the graph of g at $x = 0.3$ lies below the graph of g for $0.3 < x < 1$.	1 : answer with reason