1. Let $f(x)$ be a function defined for all $x \neq 0$ such that $f(4) = -3$ and the derivative of $f(x)$ is given by $f'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.
   (a) Find any critical points of $f(x)$ and classify the extrema. Justify your answers!
   (b) On what intervals, if any, is the graph of $f(x)$ concave up? Justify your answer!
   (c) Use the linear approximation for $f(x)$ at $x = 4$ to estimate the value of $f(3.8)$.
   (d) Is this estimate greater than or less than the actual value? Justify your answer.

$f'(x) = \frac{x^2-2}{x} = 0$, und

$x = \pm\sqrt{2}, 0$

$f'(x)$: $-$ | $+$ | $-$ | $+$

$f(x)$: $\searrow$ $-\sqrt{2}$ $\nearrow$ 0 $\searrow$ $\sqrt{2}$ $\nearrow$

a) $x = -\sqrt{2}, \sqrt{2}$ Relative min $f'(x) < 0 \to f'(x) > 0$

$x = 0$ Relative max $f'(x) > 0 \to f'(x) < 0$

b) $f''(x) = \frac{2x(x) - (x^2-2)}{x^2} = \frac{x^2+2}{x^2} = 0$, und $x^2 + 2 \neq 0$

PPOI $x = 0$

$f''(x)$: $+$ | 0 | $+$

$f(x)$: $\cup$ 0 $\cup$

concave up $(-\infty, 0) \cup (0, \infty)$ $f''(x) > 0$

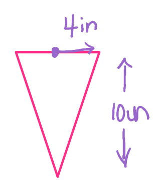
c) $f'(4) = \frac{14}{4} = \frac{7}{2}$ $y + 3 = \frac{7}{2}(x-4)$ $L(x) = \frac{7}{2}(x-4) - 3$

$f(3.8) \approx L(3.8) = \frac{7}{2}(-.2) - 3 = \frac{7}{2}\left(-\frac{1}{5}\right) - 3 = -\frac{7}{10} - \frac{30}{10} = \boxed{-3.7}$

d) Since $f''(4) > 0$, *concave up and $L(3.8)$ underestimate.

* tangent line under the curve

2. A cone is filled to the top with root beer. The height of this cone is 10 inches and the radius of its base is 4 inches. Jeff cuts a hole in the vertex so that the root beer flows out at the constant rate of 4 in³/sec allowing him to drink his root beer in one gulp. How fast is the height of the root beer inside the cone changing when the height is 5 inches? 2 inches?



$\frac{dV}{dt} = -4 \frac{in^3}{sec}$ Find $\frac{dh}{dt}$ when a) $h = 5$ in, b) $h = 2$ in.

$\frac{r}{h} = \frac{4}{10}$

$r = \frac{2}{5}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$

$\frac{d}{dt}\left(V = \frac{4\pi}{75}h^3\right)$

$\frac{dV}{dt} = \frac{4\pi}{25}h^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}$

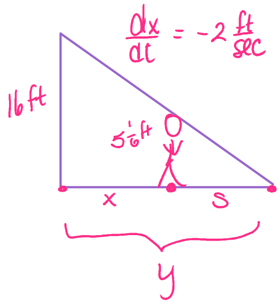
a) $h = 5$ in $\frac{dh}{dt} = \frac{\cancel{25}}{\cancel{4}\pi \cancel{(5)^2}} \cdot -\cancel{4}$

$= \boxed{\frac{-1}{\pi} \frac{in}{sec}}$

b) $h = 2$ in $\frac{dh}{dt} = \frac{25}{4\pi \cancel{(2)^2}} \cdot -\cancel{4}$

$= \boxed{\frac{-25}{4\pi} \frac{in}{sec}}$

3. All 5 ft 2 inches of Ann confidently strolls *toward* a 16ft lamppost at the constant rate of 2 feet/sec. At the instant she is 15 feet from the base of the lamppost, how fast is the length of her shadow changing? How fast is the tip of her shadow moving?



$\frac{dx}{dt} = -2 \frac{ft}{sec}$ Find a) $\frac{ds}{dt}$ when $x = 15$ ft

$\frac{16}{x+s} = \frac{\frac{31}{6}}{s}$

$6\left(16s = \frac{31}{6}x + \frac{31}{6}s\right)$

$96s = 31x + 31s$

$65s = 31x$

$\frac{d}{dt}\left(s = \frac{31}{65}x\right)$

$\frac{ds}{dt} = \frac{31}{65}\frac{dx}{dt}$

$\frac{ds}{dt} = \frac{31}{65}(-2)$

$\boxed{\frac{ds}{dt} = \frac{-62}{65} \frac{ft}{sec}}$

b) $\frac{dy}{dt}$ when $x = 15$ ft

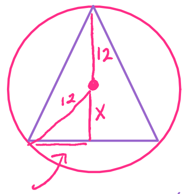
$\frac{d}{dt}(y = x + s)$

$\frac{dy}{dt} = \frac{dx}{dt} + \frac{ds}{dt}$

$\frac{dy}{dt} = -2 + \frac{-62}{65}$

$\boxed{\frac{dy}{dt} = -\frac{192}{65} \frac{ft}{sec}}$

4. Find the maximum dimensions of a right circular cone that can be inscribed in a sphere of radius 12 feet.



$r = \sqrt{144-x^2}$

$r^2 = 144 - x^2$

$h = 12 + x$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (144-x^2)(12+x)$

$V' = \frac{\pi}{3}\left[(-2x)(12+x) + 1(144-x^2)\right]$

$V' = \frac{\pi}{3}\left[-24x - 2x^2 + 144 - x^2\right]$

$V' = \frac{-\pi}{3}(3x^2 + 24x - 144)$

$V' = -\pi(x^2 + 8x - 48)$

$V' = 0 = \pi(x+12)(x-4)$

$x = \cancel{-12}, 4$

NOT in FD

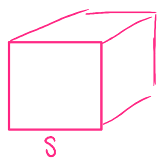
$V'' = -\pi(2x+8)$

Since $V'(4) = 0$ & $V''(4) < 0$, Volume max when $x = 4$

$h = 16$ ft

$r = \sqrt{128}$ ft $= 8\sqrt{2}$ ft

5. A cube is contracting so that its surface area decreases at the constant rate of 72 sq in/sec. How fast is the volume changing when the surface area is 54 sq ft?



$A = 6s^2$ $\qquad \frac{dA}{dt} = -72 \frac{in^2}{sec}$ $\qquad$ Find $\frac{dV}{dt}$ when $A = 54$ ft^2

$A = 54$ ft^2

$6s^2 = 54$

$s^2 = 9$

$s = \pm 3$

$s = 3$ ft

$s = 36$ in.

$\frac{d}{dt}(A = 6s^2)$

$\frac{dA}{dt} = 12s\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{1}{12s} \cdot \frac{dA}{dt}$

$\frac{ds}{dt} = \frac{1}{12(36)} \cdot -72 = -\frac{1}{6} \frac{in}{sec}$

$\frac{d}{dt}(V = s^3)$

$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$

$\frac{dV}{dt} = 3(36)^2 \cdot -\frac{1}{6}$

$\boxed{\frac{dV}{dt} = -648 \frac{in^3}{sec}}$

6. <u>Calculator Allowed</u>: Find all values of c that satisfy the MVT for the function $f(x) = x^4 - \cos(x)$ on $\left[0, \frac{\pi}{2}\right]$.

Since $f(x)$ is cont on $[0, \pi/2]$ & diff on $(0, \pi/2)$, MVT applies.

$f(0) = -1$

$f\left(\frac{\pi}{2}\right) = \frac{\pi^4}{16}$

$f'(x) = 4x^3 + \sin x$

$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} = \frac{\frac{\pi^4}{16} + 1}{\frac{\pi}{2}} = \frac{\pi^3}{8} + \frac{2}{\pi}$

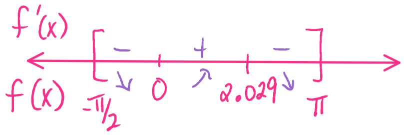
$4c^3 + \sin c = \frac{\pi^3}{8} + \frac{2}{\pi}$

using GC: $\boxed{c \approx .973}$

7. <u>Calculator Allowed</u>: Find & classify the extrema of the function $f(x) = x\sin(x)$ on $\left[-\frac{\pi}{2}, \pi\right]$.

$f'(x) = \sin x + x\cos x = 0$

$x = 0, 2.029$



$(-\pi/2, \pi/2)$ NOT extrema
≈ 1.571

$(0, 0)$ Abs min

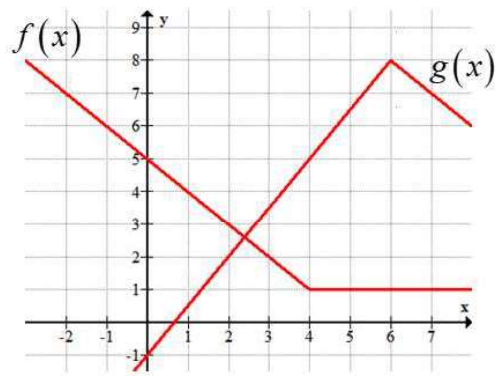
$(2.029, 1.820)$ Abs max

$(\pi, 0)$ Abs min

8. Evaluate the following: a) $\lim\limits_{x\to\infty} \frac{4}{2x^2}$ b) $\lim\limits_{x\to\infty} \frac{3x^3-4x}{10x^3}$ c) $\lim\limits_{x\to\infty} \frac{5x^4-3x}{10x^3+2x}$ d) $\lim\limits_{x\to\infty} \frac{\sin 3x}{x} = 0$

$= 0$ $\quad = \frac{3}{10}$ $\quad = \infty$ $\quad$ "finite #" / ∞

9. Use the graph to the right to evaluate each of the following:



a. $r'(-1)$ when $r(x) = (f(x))^2$

$r'(-1) = 2(f(-1)) \cdot f'(-1)$

$= 2(6)(-1) = \boxed{-12}$

b. $m'(7)$ when $m(x) = f(x)g(x)$

$m'(7) = f'(7)g(7) + f(7)g'(7)$

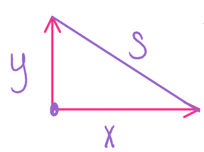
$= 0(7) + 1(-1) = \boxed{-1}$

c. $n'(4)$ when $n(x) = \ln(g(x))$

$n'(4) = \frac{1}{g(4)} \cdot g'(4)$

$= \frac{1}{5} \cdot \frac{3}{2} = \boxed{\frac{3}{10}}$

10. A balloon and a car start from the same point at the same time. The balloon is rising vertically at a constant rate of 10 ft/min while the car is traveling due west at a constant speed of 2000ft/min. How fast is the distance between the car & balloon changing 10 minutes later?



$\frac{dy}{dt} = 10\frac{ft}{min}$

$\frac{dx}{dt} = 2000\frac{ft}{min}$

find $\frac{ds}{dt}$ at $t = 10$ min

At $t = 10$min

$x = 2000\frac{ft}{min} \cdot 10$ min $\quad y = 10\frac{ft}{min} \cdot 10$min

$x = 20{,}000$ ft $\quad = 100$ ft

$S = \sqrt{(20{,}000)^2 + 100^2} = 20{,}000.25$

$\frac{d}{dt}(x^2 + y^2 = S^2)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2S\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{S}$

$\frac{ds}{dt} = \frac{20{,}000(2000) + 100(10)}{20{,}000.25}$

$\boxed{\frac{ds}{dt} = 2000.025\frac{ft}{min}}$

11. Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$

a. Find $\frac{dy}{dx}$.

b. Write an equation of each horizontal tangent line to the curve.

c. The line through the origin with slope -1 is tangent to the curve at point P. Find the x and y coordinates of point P.

a) $\frac{d}{dx}(2y^3 + 6x^2y - 12x^2 + 6y = 1)$

$6y^2\frac{dy}{dx} + 12xy + 6x^2\frac{dy}{dx} - 24x + 6\frac{dy}{dx} = 0$

$y^2\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} - 4x + \frac{dy}{dx} = 0$

$\boxed{\frac{dy}{dx} = \frac{4x - 2xy}{x^2 + y^2 + 1}}$

b) $4x - 2xy = 0$

$2x(2 - y) = 0$

$x = 0$ OR $y = 2$

$2y^3 + 6y - 1 = 0$

GC: $y = .165$

$\boxed{(0, .165)}$

$16 + 12\cancel{x^2} - 12\cancel{x^2} + 12 = 1$

$28 \neq 1$

$y = 2$ NOT on curve

c) $y = -x \quad m = -1$

$\frac{4x - 2xy}{x^2 + y^2 + 1} = -1$

$\frac{4x - 2x(-x)}{x^2 + (-x)^2 + 1} = -1$

$4x + 2x^2 = -2x^2 - 1$

$4x^2 + 4x + 1 = 0$

$(2x + 1)^2 = 0$

$x = -\frac{1}{2}$

$y = \frac{1}{2}$

$\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$

12. Evaluate the following derivatives using the limit definition of a derivative.

a. $f(x)=\dfrac{7}{3-x}$

$$f'(x)=\lim_{h\to 0}\frac{\left(\frac{7}{3-(x+h)}-\frac{7}{3-x}\right)}{h}\cdot\frac{(3-x)(3-x-h)}{(3-x)(3-x-h)}$$

$$=\lim_{h\to 0}\frac{7(3-x)-7(3-x-h)}{h(3-x)(3-x-h)}$$

$$=\lim_{h\to 0}\frac{\cancel{21}-\cancel{7x}-\cancel{21}+\cancel{7x}+7h}{h(3-x)(3-x-h)}$$

$$=\lim_{h\to 0}\frac{7\cancel{h}}{\cancel{h}(3-x)(3-x-h)}=\boxed{\frac{7}{(3-x)^2}}$$

b. $f(x)=\sqrt{2x-3}$

$$f'(x)=\lim_{h\to 0}\frac{\left(\sqrt{2(x+h)-3}-\sqrt{2x-3}\right)\left(\sqrt{2(x+h)-3}+\sqrt{2x-3}\right)}{h\left(\sqrt{2(x+h)-3}+\sqrt{2x-3}\right)}$$

$$=\lim_{h\to 0}\frac{2(x+h)-3-(2x-3)}{h\left(\sqrt{2(x+h)-3}+\sqrt{2x-3}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{2x}+2h-\cancel{3}-\cancel{2x}+\cancel{3}}{h\left(\sqrt{2(x+h)-3}+\sqrt{2x-3}\right)}$$

$$=\lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)-3}+\sqrt{2x-3}\right)}=\frac{2}{2\sqrt{2x-3}}=\boxed{\frac{1}{\sqrt{2x-3}}}$$

c. $f(x)=x^2+2x-5$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2+2(x+h)-5-(x^2+2x-5)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2+\cancel{2x}+2h-\cancel{5}-\cancel{x^2}-\cancel{2x}+\cancel{5}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h+2)}{\cancel{h}}$$

$$=\boxed{2x+2}$$

13. Let $f(x)=\begin{cases}x^2-a^2x & if \quad x<2\\ 4-2x^2 & if \quad x\ge 2\end{cases}$. Discuss the left and right sided limits as x approaches 2. Find all values of a that make the function continuous. Justify your work.

$$\lim_{x\to 2^-} f(x)=2^2-a^2(2)=4-2a^2$$

$$\lim_{x\to 2^+} f(x)=4-2(2)^2=-4$$

If f is cont at $x=2$:

① $f(2)=4-2(2)^2=-4$

② $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$

$4-2a^2=-4$

$-2a^2=-8$

$a^2=4$, $\boxed{a=\pm 2}$

③ If $a=\pm 2$, then $f(2)=-4=\lim_{x\to 2} f(x)$ and f is cont at $x=2$.

14. The graph to the right is the derivative of some function $f(x)$. Identify the intervals on which $f(x)$ is increasing/decreasing and concave up/down. Classify the x-coordinates of all extrema. Identify the x-coordinates of all points of inflection. Sketch a possible graph of $f(x)$.

$f(x)$ incr on $(-4,-3)\cup(-1,0)\cup(1,2)$
$f'(x)>0$

$f(x)$ decr on $(-3,-1)\cup(0,1)\cup(2,4)$
$f'(x)<0$

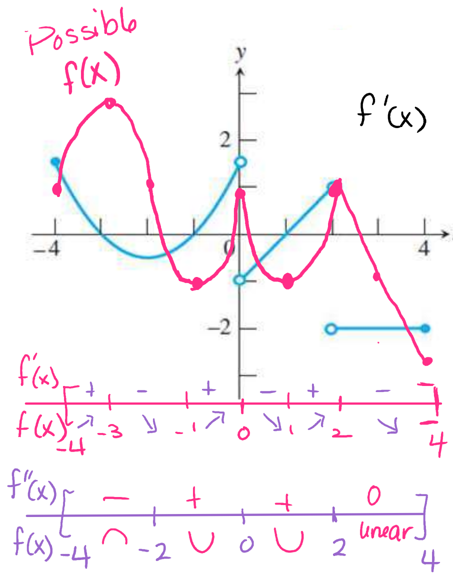
$x=-3,\ 0,\ 2$ Rel max $f'(x)>0\to f'(x)<0$

$x=-1,1$ Rel min $f'(x)<0\to f'(x)>0$

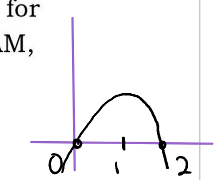
$f(x)$ concave up on $(-2,0)\cup(0,2)$
$f''(x)>0$

$f(x)$ concave down on $(-4,-2)$
$f''(x)<0$

POI $x=-2$ $f''(x)$ changes sign



15. The region **R** is enclosed between the graph of the function $y = 2x - x^2$ and the $x - axis$ for $0 \le x \le 2$. Partition [0, 2] into 4 equal subintervals and compute the LRAM, RRAM, MRAM, inscribed, circumscribed and trapezoidal approximations without a calculator.



$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

LRAM $= \frac{1}{2}\left[f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2})\right] = 1.25$

RRAM $= \frac{1}{2}\left[f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right] = 1.25$

MRAM $= \frac{1}{2}\left[f(\tfrac{1}{4}) + f(\tfrac{3}{4}) + f(\tfrac{5}{4}) + f(\tfrac{7}{4})\right] = 1.375$

Inscribed $= \frac{1}{2}\left[f(0) + f(\tfrac{1}{2}) + f(\tfrac{3}{2}) + f(2)\right] = .75$

Circum $= \frac{1}{2}\left[f(\tfrac{1}{2}) + f(1) + f(1) + f(\tfrac{3}{2})\right] = 1.75$

Trapezoid $= \frac{1}{2}\cdot\frac{1}{2}\left[f(0) + 2f(\tfrac{1}{2}) + 2f(1) + 2f(\tfrac{3}{2}) + f(2)\right] = 1.25$

16. Given $f(x) = x^5 + 2x - 1$, evaluate $\left(f^{-1}\right)'(2)$ without calculating the inverse function.

$f(x) = x^5 + 2x - 1$

$(C, 2)$

$2 = C^5 + 2C - 1$

GC: $C = 1$

$f'(x) = 5x^4 + 2$

$f'(1) = 7$

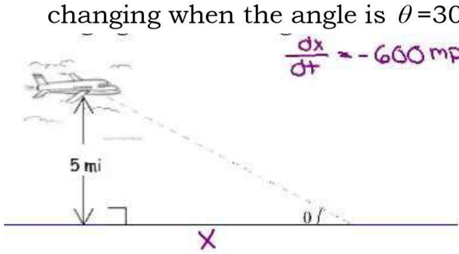
$g(x) = f^{-1}(x)$

$(2, C)$

$g'(2) = \frac{1}{f'(1)} = \boxed{\frac{1}{7}}$

17. An airplane flies at an altitude of 5 miles toward a point directly over an observer. The speed of the plane is 600 miles per hour. Find the rate at which the angle of elevation, θ, is changing when the angle is θ =30°. Find $\frac{d\theta}{dt}$ when $\theta = 30°$ $\Rightarrow \sin(30°) = \frac{1}{2}$

$\frac{dx}{dt} = -600$ mph



$\frac{d}{dt}\left(\cot\theta = \frac{x}{5}\right)$

$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-1}{5}\frac{dx}{dt} \cdot \sin^2\theta$

$\frac{d\theta}{dt} = -\frac{1}{5}(-600)\left(\frac{1}{2}\right)^2$

$\boxed{\frac{d\theta}{dt} = 30 \frac{rad}{hr}}$

18. Discuss the continuity of the following function: $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$. Refer to the definition in your justification.

To be continuous at $x = 1$,

a) $f(1) = 1$

b) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$2(1) = -1 + 3$

$2 = 2 \checkmark$

$\lim_{x \to 1} f(x) = 2$

c) Since $f(1) = 1 \neq \lim_{x \to 1} f(x) = 2$

$f(x)$ is NOT continuous at $x = 1$

There is a removable, hole discontinuity.

19. Compute $\frac{ay}{dx}$ at the point $P=(2,1)$ on the curve $y^3+3xy=7$ and show that the linear approximation at P is $L(x)=-\frac{1}{3}x+\frac{5}{3}$. Use $L(x)$ to estimate the y-coordinate of the point on the curve where $x=2.1$.

$\frac{d}{dx}(y^3+3xy=7)$

$3y^2\frac{dy}{dx}+3y+3x\frac{dy}{dx}=0$

$y^2\frac{dy}{dx}+y+x\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-y}{y^2+x}$

$\left.\frac{dy}{dx}\right|_{(2,1)}=\frac{-1}{1^2+2}=-\frac{1}{3}$

$y-1=-\frac{1}{3}(x-2)$

$L(x)=-\frac{1}{3}x+\frac{2}{3}+1=-\frac{1}{3}x+\frac{5}{3}$

$y(2.1)\approx L(2.1)=-\frac{1}{3}(.1)+1$

$=-\frac{1}{3}\left(\frac{1}{10}\right)+1=-\frac{1}{30}+1=\boxed{\frac{29}{30}}$

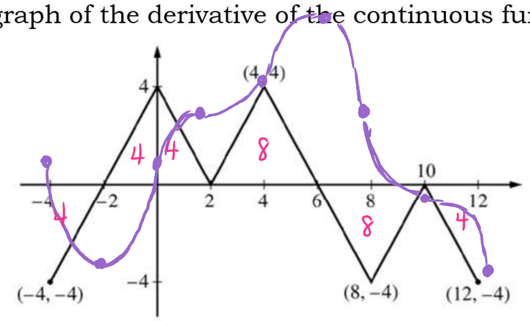
20. The graph to the right shows the graph of the derivative of the continuous function f defined on the interval $-4\le x\le 12$.

f': $-$ $+$ $+$ $-$ $-$

f: -4 ↘ -2 ↗ 2 ↗ 6 ↘ 10 ↘ 12

f'': $+$ $-$ $+$ $-$ $+$ $-$

f: -4 ∪ 0 ∩ 2 ∪ 4 ∩ 8 ∪ 10 ∩ 12



a. Does the graph of f have a relative minimum, relative maximum or neither at $x=10$? Justify your answer.

Since $f'(x)$ does not change sign at $x=10$, the graph has neither a minimum or maximum.

b. Does the graph of f have a point of inflection at $x=4$? Justify your answer.

Since $f'(x)$ changes from incr to decr ($f''(x)>0 \to f''(x)<0$) at $x=4$, the graph of f has a point of inflection.

c. On what intervals is the graph of f decreasing and concave down? Justify your answer.

$f'(x)<0$, $f'(x)$ decr ($f''(x)<0$)

$(6,8)\cup(10,12)$

d. Use the graph of f' to evaluate each of the following:

i. $\int_0^{10} f'(x)dx=0$ $\quad 4+8-8-4$

ii. $\int_6^{12} f'(x)dx=-12$ $\quad -8-4$

iii. $\int_{-4}^{0} f'(x)dx=0$ $\quad -4+4$

iv. $\int_{10}^{-2} f'(x)dx=-\int_{-2}^{10} f'(x)\,dx$

$=-[4+4+8-8]$

$=-8$

e. Sketch a possible graph of f on the graph of f'.