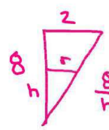
**Open-Ended Practice Problems:**

**1.** Nicole just loves drinking chocolate milk out of her special cone cup which has a radius of 2 inches and a height of 8 inches. Nicole pours milk into her cone cup at the constant rate of 2.5 in³/sec. The total amount of milk that this cone cup can hold is 33.510 in³. $\frac{dV}{dt} = 2.5 \text{ in}^3/\text{sec}$

a. Find the radius and height of the milk when the cone cup is half full.




$\frac{8}{h} = \frac{2}{r}$

$2h = 8r$

$h = 4r$

$V = \frac{\pi}{3} r^2 h$

$\frac{33.510}{2} = \frac{\pi}{3} r^2 (4r)$

$3.99996 = r^3 \Rightarrow \boxed{r \approx 1.587 \Rightarrow h \approx 6.350}$

b. How fast is the height and radius of the milk changing at the instant the cone cup is half full of milk?

$V = \frac{4\pi}{3} r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}$

$\frac{dr}{dt} = \frac{2.5}{4\pi (1.587)} = .079$

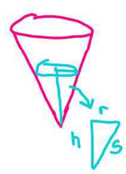
$\boxed{\frac{dr}{dt} \approx .079 \text{ in/sec}}$

$h = 4r$

$\frac{dh}{dt} = 4 \frac{dr}{dt}$

$\boxed{\frac{dh}{dt} = .316 \text{ in/sec}}$

c. The milk leaves a ring of chocolate inside the cup as it is poured. How fast is the milk ring moving up the sides of the cone cup at the instant the cone is half full?




$r^2 + h^2 = s^2$

$r^2 + (4r)^2 = s^2$

$17r^2 = s^2$

$34r \frac{dr}{dt} = 2s \frac{ds}{dt}$

$\frac{ds}{dt} = \frac{17r \frac{dr}{dt}}{s}$

$\boxed{\frac{ds}{dt} = \frac{17(1.587)(.079)}{6.545} \approx 0.326 \text{ in/sec}}$

**2.** The position equation of a particle moving along the $x$-axis is given by $x(t) = t^2 - 3t + 2$ , $t \geq 0$ .

a. Describe the motion of the particle. $v(t) = x'(t) = 2t - 3 = 0$

$t = 3/2$

$v(t)$: $-$ on $(0, 3/2)$, $+$ after $3/2$

$\boxed{\begin{array}{ll} \text{left } (0, \frac{3}{2}) & v'(t) < 0 \\ \text{right } (\frac{3}{2}, \infty) & v'(t) > 0 \end{array}}$

b. Find the position of the particle when its instantaneous velocity is zero.

$v(t) = 2t - 3 = 0$

$t = \frac{3}{2}$

$x(\frac{3}{2}) = -\frac{1}{4}$

$\boxed{\text{1/4 unit left of origin}}$

c. Find the speed of the particle when the position is zero. $x(t) = (t-2)(t-1) = 0$

$t = 2 \quad t = 1$

$v(1) = -1$

$v(2) = 1$

$\boxed{\text{speed at } t = 1 \text{ \& } t = 2 \text{ is } 1}$

d. Is the speed of the particle increasing or decreasing at $t = 3$ .

$a(t) = 2$

$\left.\begin{array}{l} v(3) = 2(3) - 3 = 3 \\ a(3) = 2 \end{array}\right\}$ $\boxed{\text{Speed is increasing since } v(0) > 0 \text{ \& } a(0) > 0}$

e. Find the average velocity of the particle during its first three seconds of travel.

$\text{avg } v = \frac{x(3) - x(0)}{3 - 0} = \frac{2 - 2}{3} = \boxed{0 \text{ units/sec}^2}$

**3.** Consider the curve given by $y^2 = 2 + xy$.

a. Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

$$2y\frac{dy}{dx} = 0 + \frac{dy}{dx}\cdot x + y$$
$$\frac{dy}{dx}(2y - x) = y$$
$$\frac{dy}{dx} = \frac{y}{2y-x}$$

b. Find all points $(x, y)$ on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

$$\frac{y}{2y-x} = \frac{1}{2}$$
$$2y = 2y - x$$
$$x = 0$$

$x = 0 \rightarrow y^2 = 2$

$y = \pm\sqrt{2}$

$\boxed{(0, \pm\sqrt{2})}$

c. Show that there are no points $(x, y)$ on the curve where the line tangent to the curve is horizontal.

$\frac{dy}{dx} = 0 \rightarrow y = 0$

$0^2 = 2 + x(0)$

$0 \neq 2$

$\therefore$ no points exist

d. Let $x$ and $y$ be functions of time $t$ that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of $y$ is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$. (No Units!)

$3^2 = 2 + x(3)$

$9 = 3x + 2$

$\frac{7}{3} = x$

$$2y\frac{dy}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$
$$2(3)(6) = 3\frac{dx}{dt} + 6\left(\frac{7}{3}\right)$$
$$36 = 3\frac{dx}{dt} + 14$$
$$\frac{22}{3} = \frac{dx}{dt}$$

$\boxed{\frac{dx}{dt} = \frac{22}{3}}$

See last page for part e!

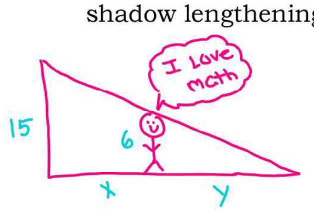
**4.** Show that the slope of every line tangent to the curve $f(x) = \frac{1}{(1-2x)^3}$ is positive.

$= (1-2x)^{-3}$

$$f'(x) = -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}$$

Since $(1-2x)$ is raised to an even power $(1-2x)^4 > 0$ and the quotient of 2 positive values is positive, $f'(x) > 0$.

**5.** A six foot tall man walks away from a 15 foot tall lamp post at 5 ft/sec. How fast is his shadow lengthening? How fast is the shadow's tip moving?




A. $\frac{15}{6} = \frac{x+y}{y}$

$6x + 6y = 15y$

$9y = 6x$

$\frac{d}{dt}\left(y = \frac{2}{3}x\right)$

$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{2}{3}(5) = \boxed{\frac{10}{3}\text{ ft/sec}}$

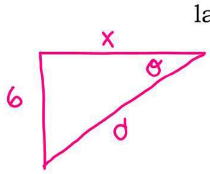
B. $\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$

$= 5 + \frac{10}{3}$

$\boxed{= \frac{25}{3}\text{ ft/sec}}$

**6.** At a given moment, a plane passes directly above a radar station at an altitude of 6 km. The plane's speed is 800 km/h. $\frac{dx}{dt} = 800 \text{ km/h}$

a. How fast is the distance between the plane and the station changing half a minute later?




$x^2 + 6^2 = d^2$

$2x\frac{dx}{dt} = 2d\frac{dd}{dt}$

$\frac{dd}{dt} = \frac{x}{d}\left(\frac{dx}{dt}\right)$

$\frac{dd}{dt} = \frac{\frac{20}{3}}{\frac{2\sqrt{181}}{3}}(800) = \frac{10}{\sqrt{181}} \cdot 800 = \boxed{\frac{8000}{\sqrt{181}} \text{ km/h}}$

$t = \frac{1}{2}\text{min} = \frac{1}{120}\text{hr}$

$\hookrightarrow x = \frac{20}{3}$ km

$6^2 + \left(\frac{20}{3}\right)^2 = d^2$

$d^2 = \frac{724}{9}$

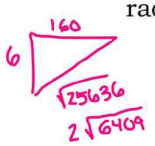
$d = \frac{2\sqrt{181}}{3}$

b. How fast is the distance between the plane and the station changing when the plane passes directly above the station?

$\frac{dd}{dt} = \frac{0}{6}(800) = \boxed{0 \text{ km/hr}}$

c. Let θ be the angle that the line through the radar station and the plane makes with the horizontal. How fast is θ changing 12 minutes after the plane passes over the radar station?

$\underbrace{\text{12 minutes}}_{\frac{1}{5}\text{hr}}$ $\rightarrow x = \frac{800}{5} = 160$




$\tan\theta = \frac{6}{x}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{-6}{x^2}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-6\cos^2\theta}{x^2}\frac{dx}{dt}$

$= \frac{-6\left(\frac{160}{2\sqrt{6409}}\right)^2}{160^2} \cdot 800 = \frac{-6\left(\frac{160^2}{4(6409)}\right)}{160^2} \cdot 800 = \frac{-3}{2} \cdot \frac{800}{6409} = \frac{-1200}{6409} \text{ rad/hr}$

$\boxed{\frac{-1200}{6409} \text{ rad/hr}}$

**7.** Let $s(t) = t^3 - 6t^2 + 9t$ be the position function, in feet, for a projectile's path along the $y$-axis for $t \geq 0$, for $t$ seconds.

$s'(t) = v(t) = 3t^2 - 12t + 9$

$s''(t) = a(t) = 6t - 12$

a. What is the acceleration of the particle at 8 seconds? $a(8) = 6(8) - 12 = \boxed{36 \text{ ft/sec}^3}$

b. What is the average velocity of the particle during the first four seconds?

$\text{avg } v = \frac{s(4) - s(0)}{4 - 0} = \frac{4}{4} = \boxed{1 \text{ ft/sec}}$

c. Describe the motion of the particle.

$v(t) = 3(t^2 - 4t + 3) = 0$

$3(t-3)(t-1) = 0$

$v(t)$: + on $(0,1)$, − on $(1,3)$, + after 3

$\boxed{\text{moving up } (0,1) \cup (3,\infty) \quad \text{moving down } (1,3)}$

d. What is the speed of the particle when the acceleration is zero?

$a(t) = 6t - 12 = 0$

$t = 2$

$v(2) = 3(2)^2 - 12(2) + 9 = -3$

$\text{Speed} = |v(2)| = \boxed{3 \text{ ft/sec}}$

e. What is the displacement of the particle during the first 3 seconds?

$s(0) = 0 \qquad s(3) = 0$

$\boxed{0 \text{ ft}}$

f. Is the speed of the particle increasing or decreasing when $t = 8$?

$v(8) = 3(8)^2 - 12(8) + 9 = 105$

$a(8) = 6(8) - 12 = 36$

$\boxed{\text{Since } v(8) > 0 \text{ and } a(8) > 0 \text{ Speed is increasing.}}$

**8.** Sand is falling from a rectangular box container whose base measures 40 inches by 20 inches at a constant rate of 300 cubic inches per minute. $\frac{dV}{dt} = -300 \text{ in}^3/\text{min}$

a. How fast is the depth of the sand in the box changing? *(Exact answers only)*

$V = 40(20)h$

$\frac{dV}{dt} = 800\frac{dh}{dt}$

$\boxed{-\frac{3}{8} \text{ in/min} = \frac{dh}{dt}}$

b. As the sand falls from the rectangular box it forms a conical pile on the ground. At a particular moment, the pile is 23 inches high and the diameter of the base is 16 inches. The radius of the base at this moment is increasing at $\frac{3}{4}$ inches per minute.

$r = 8$

$\frac{dr}{dt} = \frac{3}{4}$

i. At this moment, how fast is the area of the circular base of the cone increasing? *(Exact answers only)*

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(8)(\frac{3}{4}) = \boxed{12\pi \text{ in}^2/\text{min}}$

ii. Write an equation for the rate of change of the volume in terms of the radius and height. Determine how fast the height of the pile increasing at this moment. *(Round your answer to nearest thousandth.)*

$V = \frac{1}{3}\pi r^2 h$

$\boxed{\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dr}{dt} h + \frac{\pi}{3} r^2 \frac{dh}{dt}}$

$300 = \frac{2\pi}{3} \cdot 8 \cdot \frac{3}{4} \cdot 23 + \frac{\pi}{3}(8)^2 \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = .164 \text{ in/min}}$

3. e. Find all points $(x, y)$ on the graph, if any, where the curve will have a vertical tangent line.

$$y^2 = 2 + xy$$

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

$2y - x = 0$

$2y = x$

$y^2 = 2 + 2y^2$

$-y^2 = 2$

$y^2 = -2$ No real solution $\Rightarrow$ No vert. tan. lines!