1. Evaluate the indefinite or definite integral. Check when applicable on the calculator.

a. $\int_0^{\frac{\pi}{2}} 2\sin x\sqrt{\cos x}\,dx$

$u=\cos x$

$\frac{du}{dx}=-\sin x$

$dx=\frac{du}{-\sin x}$

$x=0, u=1$

$x=\pi/2, u=0$

$= 2\int_1^0 \cancel{\sin x}\sqrt{u}\,\frac{du}{-\cancel{\sin x}}$

$= 2\int_0^1 u^{1/2}du$

$= 2\left(\frac{2}{3}u^{3/2}\right)_0^1$

$= \frac{4}{3}(1-0) = \boxed{\frac{4}{3}}$

b. $\int \frac{x+3}{\sqrt{x-2}}dx = \int \frac{u+2+3}{\sqrt{u}}du$

$u=x-2$

$x=u+2$

$du=dx$

$= \int \frac{u+5}{\sqrt{u}}du$

$= \int (u^{1/2}+5u^{-1/2})du$

$= \frac{2}{3}u^{3/2}+10u^{1/2}+C$

$= \boxed{\frac{2}{3}(x-2)^{3/2}+10(x-2)^{1/2}+C}$

c. $\int \frac{t^2}{(16-t^3)^2}dt$

$u=16-t^3$

$\frac{du}{dt}=-3t^2$

$dt=\frac{du}{-3t^2}$

$\int \frac{\cancel{t^2}}{u^2}\cdot\frac{du}{-3\cancel{t^2}}$

$= -\frac{1}{3}\int u^{-2}du$

$= -\frac{1}{3}(-u^{-1})+C$

$= \boxed{\frac{1}{3(16-t^3)}+C}$

2. Use the properties of even and odd functions to answer the following. Justify your answer.
Remember: $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

a. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\cos x + \sin x\cos x\,dx = 0$ *

$f(x) = \sin^3 x\cos x + \sin x\cos x$

$f(-x) = \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x)$

$= -\sin^3 x\cos x - \sin x\cos x$

$f(-x) = -f(x) \therefore f(x)$ is ODD *

b. If $\int_0^7 f(x)dx = \frac{5}{7}$ and $\int_0^{21} f(x)dx = 5$, where $f(x)$ is an odd function, find the following.

i. $\int_{-7}^0 f(x)dx = -\frac{5}{7}$

ii. $\int_{-7}^7 f(x)dx = 0$

iii. $\int_{-7}^0 -3f(x)dx =$

$= -3\int_{-7}^0 f(x)dx = -3\cdot -\frac{5}{7} = \boxed{\frac{15}{7}}$

iv. $\int_2^2 f(x)dx = 0$

v. $\int_{-7}^0 f(3x)dx =$

$u=3x$

$\frac{du}{dx}=3$

$dx=\frac{du}{3}$

$x=-7, u=-21$

$x=0, u=0$

$= \frac{1}{3}\int_{-21}^0 f(u)du = \frac{1}{3}(-5)$

$= \boxed{-\frac{5}{3}}$

3. Find the average value of the function on the given interval.

a. $y=\frac{1}{x}$ on $[e, 2e]$

$Avg = \frac{1}{2e-e}\int_e^{2e}\frac{1}{x}dx$

$= \frac{1}{e}(\ln x)_e^{2e}$

$= \frac{1}{e}[\ln(2e) - \ln e]$

$= \frac{1}{e}[\ln(2) + \cancel{\ln e} - \cancel{\ln e}] = \boxed{\frac{\ln 2}{e}}$

b. $y=\frac{1}{1+x^2}$ on $[0,1]$

$Avg = \frac{1}{1-0}\int_0^1 \frac{1}{1+x^2}dx$

$= \arctan x\Big|_0^1$

$= \arctan(1) - \arctan(0)$

$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$

4. Find $\frac{dy}{dx}$.

a. $\frac{d}{dx}\left(y = \int_{-2}^{x} \sqrt{1+e^{5t}}\,dt \right)$

$\boxed{\frac{dy}{dx} = \sqrt{1+e^{5x}}}$

b. $y = \int_{x^2}^{3} \cot(3t)\,dt = -\int_{3}^{x^2} \cot(3t)\,dt$

$\frac{dy}{dx} = -\cot(3x^2)\cdot 2x$

$= \boxed{-2x\cot(3x^2)}$

c. $y = \int_{\sin x}^{\cos x} t^2\,dt = \int_{\sin x}^{a} t^2\,dt + \int_{a}^{\cos x} t^2\,dt$

$= -\int_{a}^{\sin x} t^2\,dt + \int_{a}^{\cos x} t^2\,dt$

$= -\sin^2 x\cos x + \cos^2 x(-\sin x)$

$= \boxed{-\sin^2 x\cos x - \cos^2 x\sin x}$

5. Use the function $h(x) = \int_3^x \sqrt{t+1}\,dt$ to find the indicated value.

a. Find $h'(8)$.

$h'(x) = \sqrt{x+1}$

$\boxed{h'(8) = 3}$

b. Find the tangent line to $h(x)$ at $x = 8$.

$h(8) = \int_3^8 (t+1)^{1/2}\,dt = \frac{2}{3}(t+1)^{3/2}\Big|_3^8 = \frac{2}{3}\left[9^{3/2} - 4^{3/2}\right]$

$h(8) = \frac{2}{3}(27-8) = \frac{38}{3}$

$\boxed{y - \frac{38}{3} = 3(x-8)}$

6. Solve the following differential equations with the given initial conditions.

a. $\frac{dy}{dx} = \frac{x^3 - 2x^2 + 1}{x^2}$ with initial condition $(1,3)$

$\int dy = \int (x - 2 + x^{-2})\,dx$

$y = \frac{x^2}{2} - 2x - \frac{1}{x} + C$

$(1,3)$ $\quad 3 = \frac{1}{2} - 2 - 1 + C$

$\frac{11}{2} = 5\frac{1}{2} = C$

$\boxed{y = \frac{x^2}{2} - 2x - \frac{1}{x} + \frac{11}{2}}$

b. $\frac{dy}{dx} = \sqrt{x} + \cos x - \sin x$ with initial condition (0,0)

$\int dy = \int (x^{1/2} + \cos x - \sin x)\,dx$

$y = \frac{2}{3}x^{3/2} + \sin x + \cos x + C$

$(0,0)$

$0 = 1 + C$

$C = -1$

$\boxed{y = \frac{2}{3}x^{3/2} + \sin x + \cos x - 1}$

$f'(x) < 0$

7. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Let $h(x) = \int_0^x f(t)dt$.

Determine if the following statements are true or false. Justify your answers!

a. h is a twice-differentiable function of x.

True! $h'(x) = f(x)$, $h''(x) = f'(x)$

b. h and $\frac{dh}{dx}$ are both continuous. True! If $f(x)$ is diff → continuous

since $h'(x) = f(x)$, both h & h' are also cont.

c. The graph of h has a horizontal tangent at $x = 1$. True

$h'(1) = f(1) = 0$

d. h has a local maximum at $x = 1$. True

$h''(x) = f'(x) < 0$ concave down, Since $h'(1) = 0$ & $h''(1) < 0$

Rel max

e. h has a local minimum at $x = 1$.

false - see d.

f. The graph of h has a point of inflection at $x = 1$.

false Since $h''(x) = f'(x) < 0$ for all x, always concave down.

g. The graph of $\frac{dh}{dx}$ crosses the x-axis at $x = 1$.

True $h'(1) = f(1) = 0$, since $h''(x) < 0$, $h'(x)$ is always decreasing - MUST pass through xaxis!

8.

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$.

a. Use a right Riemann sum with five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)dt$ in terms of the radius of the balloon.

$\int_0^{12} r'(t)dt = 2(4) + 3(2) + 2(1.2) + 4(.6) + 1(.5) = 19.3$ feet

The radius of the balloon expands 19.3 ft over the 12 min.

b. Is your approximation in part (a) greater than or less than $\int_0^{12} r'(t)dt$? Give a reason for your answer. Since r is concave down, $r'(t)$ is decreasing.

∴ Rectangles are below curve → less than actual.

9. If $F(x) = \int_{1}^{x} (3t+4)\,dt$, find the extrema of $F(x)$ and the intervals on which $F(x)$ increases and decreases. Identify intervals where $F(x)$ is concave up and concave down. Justify all your answers.

$F'(x) = 3x+4 = 0$

$x = -\frac{4}{3}$

$F'(x)$ $-$ $+$

$F(x)$ $\searrow$ $-4/3$ $\nearrow$

Decr on $(-\infty, -4/3)$, $F'(x) < 0$

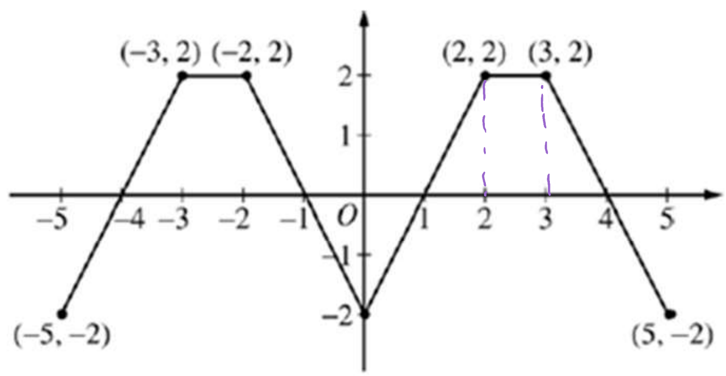
Incr on $(-4/3, \infty)$, $F'(x) > 0$

$*(-4/3, -49/6)$ absolute min

$F''(x) = 3 > 0 \Rightarrow F(x)$ concave up on $(-\infty, \infty)$ since $F''(x) > 0$

$* F(-4/3) = \int_{1}^{-4/3} (3t+4)\,dt = \frac{-49}{6}$

10.



Graph of f

The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

a. Find $g(4)$, $g'(4)$, and $g''(4)$.

$g(4) = \int_0^4 f(t)\,dt = -\frac{1}{2}(2)(1) + \frac{1}{2}(2)(1+3) = -1 + 4 = 3$

$g'(4) = f(4) = 0$ $\quad g''(4) = f'(4) = -2$

b. Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

Since $g'(x) = f(x)$, $g'(x) < 0 \rightarrow g'(x) > 0$

$g(x)$ has relative min

c. Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

Since periodic, $g(10) = 2(g(5)) = 4$

$g(3) = \int_0^3 f(t)\,dt = -\frac{1}{2}(2)(1) + \frac{1}{2}(2)(1+2)$

$= -1 + 3 = 2$

$g(108) = 21\underbrace{(g(5))}_{105} + g(3) = 21(2) + 2$

$= 44$

$g'(108) = g'(3) = 2$

$\boxed{y - 44 = 2(x - 108)}$

11. Consider the function $f(x)=\sqrt{x}$.

a. Find the area enclosed by the function & the x-axis over the interval [1, 4].

$$\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left[4^{3/2}-1^{3/2}\right] = \frac{2}{3}(8-1) = \boxed{\frac{14}{3}}$$

b. Find the average value over this interval.

$$Avf = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\cdot\frac{14}{3} = \boxed{\frac{14}{9}}$$

c. Find the average rate of change of the function on this interval.

$$\frac{f(4)-f(1)}{4-1} = \frac{2-1}{3} = \boxed{\frac{1}{3}}$$

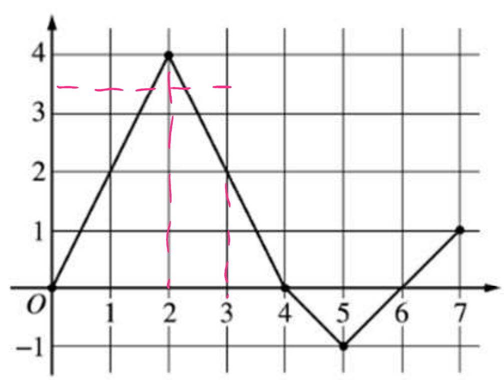
d. Find the value of c that satisfies the MVT for integrals over this interval.

$f(c) = Avf$ $\quad \boxed{c=\frac{196}{81}}$

$\sqrt{c} = \frac{14}{9}$

COVER THE ANSWERS BELOW BEFORE BEGINNING THE NEXT PROBLEM!
THEN SCORE YOURSELF ON THIS PROBLEM.

12.



Graph of f

a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(1)(4+2)$

$g(3) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = -2$

b) $g(0) = \int_2^0 f(t)\,dt = -\int_0^2 f(t)\,dt$

$= -\frac{1}{2}(2)(4) = -4$

Let f be a function defined on the closed interval [0, 7]. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x)=\int_2^x f(t)\,dt$.

b) $\frac{g(3)-g(0)}{3-0} = \frac{3-(-4)}{3} = \frac{7}{3}$

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning. $g'(c) = f(c) = \frac{7}{3}$ $\quad f(c)=\frac{7}{3}$ for two values on $(0,3)$

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

POI occur when $g''(x)$ changes sign or $g'(x)$ changes incr $\rightleftarrows$ decr

$g'(x) = f(x)$ changes from incr → decr at $x=2$ and decr → incr at $x=5$.

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$$

Note: 1/2 if answer is 1 by MVT

(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad \text{(ignore discussion at } x = 4) \end{cases}$$