

CHAPTER 4

Integration

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CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

$$1. \frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$$

$$2. \frac{d}{dx}\left(x^4 + \frac{1}{x} + C\right) = 4x^3 - \frac{1}{x^2}$$

$$3. \frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$$

$$4. \frac{d}{dx}\left(\frac{2(x^2 + 3)}{3\sqrt{x}} + C\right) = \frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C\right) \\ = x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$$

$$5. \frac{dy}{dt} = 3t^2 \\ y = t^3 + C$$

Check: $\frac{d}{dt}[t^3 + C] = 3t^2$

$$6. \frac{dr}{d\theta} = \pi \\ r = \pi\theta + C$$

Check: $\frac{d}{d\theta}[\pi\theta + C] = \pi$

$$7. \frac{dy}{dx} = x^{3/2} \\ y = \frac{2}{5}x^{5/2} + C$$

Check: $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$

$$8. \frac{dy}{dx} = 2x^{-3} \\ y = \frac{2x^{-2}}{-2} + C = \frac{-1}{x^2} + C$$

Check: $\frac{d}{dx}\left[\frac{-1}{x^2} + C\right] = 2x^{-3}$

<i>Given</i>	<i>Rewrite</i>	<i>Integrate</i>	<i>Simplify</i>
9. $\int \sqrt[3]{x} dx$	$\int x^{1/3} dx$	$\frac{x^{4/3}}{4/3} + C$	$\frac{3}{4}x^{4/3} + C$
10. $\int \frac{1}{x^2} dx$	$\int x^{-2} dx$	$\frac{x^{-1}}{-1} + C$	$-\frac{1}{x} + C$
11. $\int \frac{1}{x\sqrt{x}} dx$	$\int x^{-3/2} dx$	$\frac{x^{-1/2}}{-1/2} + C$	$-\frac{2}{\sqrt{x}} + C$
12. $\int x(x^2 + 3) dx$	$\int (x^3 + 3x) dx$	$\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$	$\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$
13. $\int \frac{1}{2x^3} dx$	$\frac{1}{2} \int x^{-3} dx$	$\frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C$	$-\frac{1}{4x^2} + C$
14. $\int \frac{1}{(3x)^2} dx$	$\frac{1}{9} \int x^{-2} dx$	$\frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{-1}{9x} + C$

$$15. \int (x + 3) dx = \frac{x^2}{2} + 3x + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{x^2}{2} + 3x + C \right] = x + 3$$

$$17. \int (2x - 3x^2) dx = x^2 - x^3 + C$$

$$\text{Check: } \frac{d}{dx} [x^2 - x^3 + C] = 2x - 3x^2$$

$$19. \int (x^3 + 2) dx = \frac{1}{4}x^4 + 2x + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{1}{4}x^4 + 2x + C \right) = x^3 + 2$$

$$21. \int (x^{3/2} + 2x + 1) dx = \frac{2}{5}x^{5/2} + x^2 + x + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{2}{5}x^{5/2} + x^2 + x + C \right) = x^{3/2} + 2x + 1$$

$$23. \int \sqrt[3]{x^2} dx = \int x^{2/3} dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{3}{5}x^{5/3} + C \right) = x^{2/3} = \sqrt[3]{x^2}$$

$$25. \int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\text{Check: } \frac{d}{dx} \left(-\frac{1}{2x^2} + C \right) = \frac{1}{x^3}$$

$$27. \int \frac{x^2 + x + 1}{\sqrt{x}} dx = \int (x^{3/2} + x^{1/2} + x^{-1/2}) dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C \right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$$

$$28. \int \frac{x^2 + 2x - 3}{x^4} dx = \int (x^{-2} + 2x^{-3} - 3x^{-4}) dx = \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C \right] = x^{-2} + 2x^{-3} - 3x^{-4} = \frac{x^2 + 2x - 3}{x^4}$$

$$16. \int (5 - x) dx = 5x - \frac{x^2}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[5x - \frac{x^2}{2} + C \right] = 5 - x$$

$$18. \int (4x^3 + 6x^2 - 1) dx = x^4 + 2x^3 - x + C$$

$$\text{Check: } \frac{d}{dx} [x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$$

$$20. \int (x^3 - 4x + 2) dx = \frac{x^4}{4} - 2x^2 + 2x + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{x^4}{4} - 2x^2 + 2x + C \right] = x^3 - 4x + 2$$

$$22. \int \left(\sqrt{x} + \frac{1}{2\sqrt{x}} \right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2} \right) dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C$$

$$= \frac{2}{3}x^{3/2} + x^{1/2} + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{2}{3}x^{3/2} + x^{1/2} + C \right) = x^{1/2} + \frac{1}{2}x^{-1/2}$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}$$

$$24. \int (\sqrt[4]{x^3} + 1) dx = \int (x^{3/4} + 1) dx = \frac{4}{7}x^{7/4} + x + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{4}{7}x^{7/4} + x + C \right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$$

$$26. \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left(-\frac{1}{3x^3} + C \right) = \frac{1}{x^4}$$

$$29. \int (x+1)(3x-2) dx = \int (3x^2 + x - 2) dx$$

$$= x^3 + \frac{1}{2}x^2 - 2x + C$$

Check: $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$$= (x+1)(3x-2)$$

$$31. \int y^2 \sqrt{y} dy = \int y^{5/2} dy = \frac{2}{7}y^{7/2} + C$$

Check: $\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2 \sqrt{y}$

$$33. \int dx = \int 1 dx = x + C$$

Check: $\frac{d}{dx}(x + C) = 1$

$$35. \int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + C$$

Check: $\frac{d}{dx}(-2 \cos x + 3 \sin x + C) = 2 \sin x + 3 \cos x$

$$37. \int (1 - \csc t \cot t) dt = t + \csc t + C$$

Check: $\frac{d}{dt}(t + \csc t + C) = 1 - \csc t \cot t$

$$39. \int (\sec^2 \theta - \sin \theta) d\theta = \tan \theta + \cos \theta + C$$

Check: $\frac{d}{d\theta}(\tan \theta + \cos \theta + C) = \sec^2 \theta - \sin \theta$

$$41. \int (\tan^2 y + 1) dy = \int \sec^2 y dy = \tan y + C$$

Check: $\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

$$30. \int (2t^2 - 1)^2 dt = \int (4t^4 - 4t^2 + 1) dt$$

$$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$$

Check: $\frac{d}{dt}\left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C\right) = 4t^4 - 4t^2 + 1$

$$= (2t^2 - 1)^2$$

$$32. \int (1 + 3t)t^2 dt = \int (t^2 + 3t^3) dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 = (1 + 3t)t^2$

$$34. \int 3 dt = 3t + C$$

Check: $\frac{d}{dt}(3t + C) = 3$

$$36. \int (t^2 - \sin t) dt = \frac{1}{3}t^3 + \cos t + C$$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \cos t + C\right) = t^2 - \sin t$

$$38. \int (\theta^2 + \sec^2 \theta) d\theta = \frac{1}{3}\theta^3 + \tan \theta + C$$

Check: $\frac{d}{d\theta}\left(\frac{1}{3}\theta^3 + \tan \theta + C\right) = \theta^2 + \sec^2 \theta$

$$40. \int \sec y(\tan y - \sec y) dy = \int (\sec y \tan y - \sec^2 y) dy$$

$$= \sec y - \tan y + C$$

Check: $\frac{d}{dy}(\sec y - \tan y + C) = \sec y \tan y - \sec^2 y$

$$= \sec y(\tan y - \sec y)$$

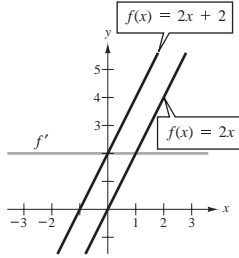
$$42. \int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = \int \left(\frac{1}{\sin x}\right) \left(\frac{\cos x}{\sin x}\right) dx$$

$$= \int \csc x \cot x dx = -\csc x + C$$

Check: $\frac{d}{dx}[-\csc x + C] = \csc x \cot x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$

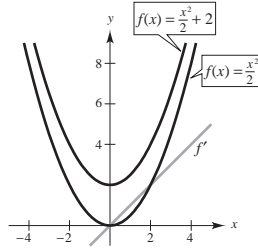
$$= \frac{\cos x}{1 - \cos^2 x}$$

43. $f'(x) = 2$
 $f(x) = 2x + C$

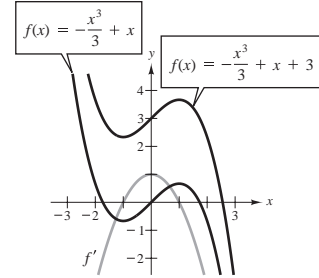


Answers will vary.

44. $f'(x) = x$
 $f(x) = \frac{x^2}{2} + C$

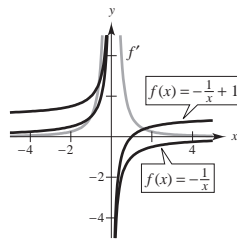


45. $f'(x) = 1 - x^2$
 $f(x) = x - \frac{x^3}{3} + C$



Answers will vary.

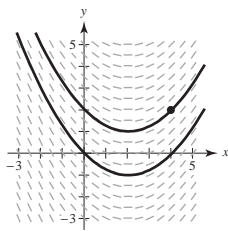
46. $f'(x) = \frac{1}{x^2}$
 $f(x) = -\frac{1}{x} + C$



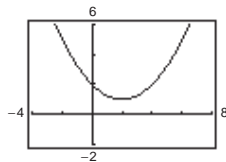
47. $\frac{dy}{dx} = 2x - 1, (1, 1)$
 $y = \int (2x - 1) dx = x^2 - x + C$
 $1 = (1)^2 - (1) + C \Rightarrow C = 1$
 $y = x^2 - x + 1$

48. $\frac{dy}{dx} = 2(x - 1) = 2x - 2, (3, 2)$
 $y = \int 2(x - 1) dx = x^2 - 2x + C$
 $2 = (3)^2 - 2(3) + C \Rightarrow C = -1$
 $y = x^2 - 2x - 1$

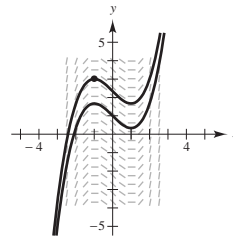
49. (a) Answers will vary.



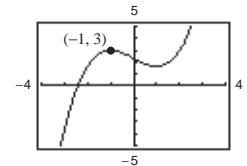
(b) $\frac{dy}{dx} = \frac{1}{2}x - 1, (4, 2)$
 $y = \frac{x^2}{4} - x + C$
 $2 = \frac{4^2}{4} - 4 + C$
 $2 = C$
 $y = \frac{x^2}{4} - x + 2$



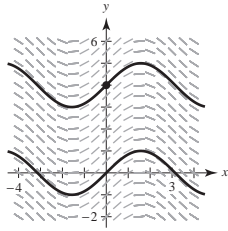
50. (a)



(b) $\frac{dy}{dx} = x^2 - 1, (-1, 3)$
 $y = \frac{x^3}{3} - x + C$
 $3 = \frac{(-1)^3}{3} - (-1) + C$
 $3 = -\frac{1}{3} + 1 + C$
 $C = \frac{7}{3}$
 $y = \frac{x^3}{3} - x + \frac{7}{3}$



51. (a)

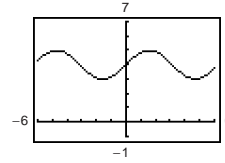


(b) $\frac{dy}{dx} = \cos x, (0, 4)$

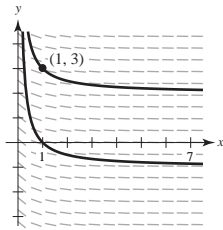
$$y = \int \cos x \, dx = \sin x + C$$

$$4 = \sin(0) + C \Rightarrow C = 4$$

$$y = \sin x + 4$$



52. (a)

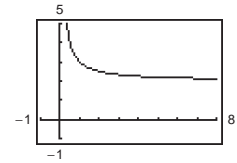


(b) $\frac{dy}{dx} = \frac{-1}{x^2}, x > 0, (1, 3)$

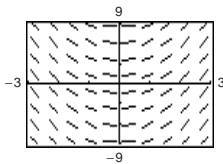
$$y = \int -\frac{1}{x^2} \, dx = \int -x^{-2} \, dx = \frac{-x^{-1}}{-1} + C = \frac{1}{x} + C$$

$$3 = \frac{1}{1} + C \Rightarrow C = 2$$

$$y = \frac{1}{x} + 2$$



53. (a)



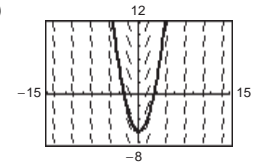
(b) $\frac{dy}{dx} = 2x, (-2, -2)$

$$y = \int 2x \, dx = x^2 + C$$

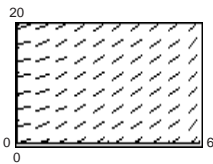
$$-2 = (-2)^2 + C = 4 + C \Rightarrow C = -6$$

$$y = x^2 - 6$$

(c)



54. (a)



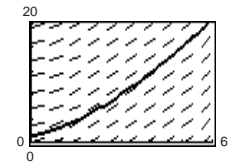
(b) $\frac{dy}{dx} = 2\sqrt{x}, (4, 12)$

$$y = \int 2x^{1/2} \, dx = \frac{4}{3}x^{3/2} + C$$

$$12 = \frac{4}{3}(4)^{3/2} + C = \frac{4}{3}(8) + C = \frac{32}{3} + C \Rightarrow C = \frac{4}{3}$$

$$y = \frac{4}{3}x^{3/2} + \frac{4}{3}$$

(c)



55. $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x \, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

56. $g'(x) = 6x^2, g(0) = -1$

$$g(x) = \int 6x^2 \, dx = 2x^3 + C$$

$$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1$$

$$g(x) = 2x^3 - 1$$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5) \, dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

58. $f'(s) = 6s - 8s^3, f(2) = 3$

$$f(s) = \int (6s - 8s^3) \, ds = 3s^2 - 2s^4 + C$$

$$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \Rightarrow C = 23$$

$$f(s) = 3s^2 - 2s^4 + 23$$

59. $f''(x) = 2$

$f'(2) = 5$

$f(2) = 10$

$f'(x) = \int 2 dx = 2x + C_1$

$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$

$f'(x) = 2x + 1$

$f(x) = \int (2x + 1) dx = x^2 + x + C_2$

$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$

$f(x) = x^2 + x + 4$

60. $f''(x) = x^2$

$f'(0) = 6$

$f(0) = 3$

$f'(x) = \int x^2 dx = \frac{1}{3}x^3 + C_1$

$f'(0) = 0 + C_1 = 6 \Rightarrow C_1 = 6$

$f'(x) = \frac{1}{3}x^3 + 6$

$f(x) = \int \left(\frac{1}{3}x^3 + 6\right) dx = \frac{1}{12}x^4 + 6x + C_2$

$f(0) = 0 + 0 + C_2 = 3 \Rightarrow C_2 = 3$

$f(x) = \frac{1}{12}x^4 + 6x + 3$

61. $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$f'(x) = \int x^{-3/2} dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$

$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$

$f'(x) = -\frac{2}{\sqrt{x}} + 3$

$f(x) = \int (-2x^{-1/2} + 3) dx = -4x^{1/2} + 3x + C_2$

$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$

$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$

62. $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \int \sin x dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$

$f'(x) = -\cos x + 2$

$f(x) = \int (-\cos x + 2) dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$

$f(x) = -\sin x + 2x + 6$

63. (a) $h(t) = \int (1.5t + 5) dt = 0.75t^2 + 5t + C$

$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$

$h(t) = 0.75t^2 + 5t + 12$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

64. $\frac{dP}{dt} = k\sqrt{t}$, $0 \leq t \leq 10$

$P(t) = \int kt^{1/2} dt = \frac{2}{3}kt^{3/2} + C$

$P(0) = 0 + C = 500 \Rightarrow C = 500$

$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$

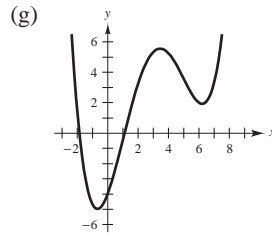
$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$

$P(7) = 100(7)^{3/2} + 500 \approx 2352$ bacteria

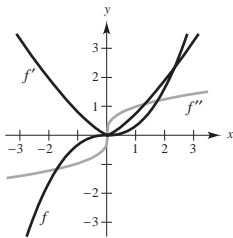
65. $f(0) = -4$. Graph of f' is given.

- (a) $f'(4) \approx -1.0$
 (b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.
 (c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.
 (d) f is a maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the First Derivative Test.

- (e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.
 (f) f'' is a minimum at $x = 3$.



66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



67. $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32 \, dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60) \, dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$$s(t) = -16t^2 + 60t + 6, \text{ Position function}$$

The ball reaches its maximum height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds.}$$

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 = 62.25 \text{ feet}$$

68. $f''(t) = a(t) = -32 \text{ ft/sec}^2$

$$f'(0) = v_0$$

$$f(0) = s_0$$

$$f'(t) = v(t) = \int -32 \, dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \Rightarrow C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0) \, dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \Rightarrow C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

69. From Exercise 68, we have:

$$s(t) = -16t^2 + v_0t$$

$s'(t) = -32t + v_0 = 0$ when $t = \frac{v_0}{32} =$ time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35,200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

70. $v_0 = 16$ ft/sec

$$s_0 = 64 \text{ ft}$$

(a) $s(t) = -16t^2 + 16t + 64 = 0$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $v(t) = s'(t) = -32t + 16$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

71. $a(t) = -9.8$

$$v(t) = \int -9.8 dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0) dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$$

72. From Exercise 71, $f(t) = -4.9t^2 + 1800$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1800$$

$$4.9t^2 = 1800$$

$$t^2 = \frac{1800}{4.9} \Rightarrow t \approx 9.2 \text{ sec}$$

73. From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$$v(t) = -9.8t + 10 = 0 \text{ (Maximum height when } v = 0.)$$

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

74. From Exercise 71, $f(t) = -4.9t^2 + v_0t + 2$. If

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9 v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9 v_0^2 + 9.8 v_0^2 = (9.8)^2 198$$

$$4.9 v_0^2 = (9.8)^2 198$$

$$v_0^2 = 3880.8 \Rightarrow v_0 \approx 62.3 \text{ m/sec.}$$

75. $a = -1.6$

$$v(t) = \int -1.6 dt = -1.6t + v_0 = -1.6t, \text{ since the stone was dropped, } v_0 = 0.$$

$$s(t) = \int (-1.6t) dt = -0.8t^2 + s_0$$

$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

$$76. \int v \, dv = -GM \int \frac{1}{y^2} \, dy$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

$$78. x(t) = (t-1)(t-3)^2 \quad 0 \leq t \leq 5$$

$$= t^3 - 7t^2 + 15t - 9$$

$$(a) v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$$

$$a(t) = v'(t) = 6t - 14$$

$$(b) v(t) > 0 \text{ when } 0 < t < \frac{5}{3} \text{ and } 3 < t < 5.$$

$$(c) a(t) = 6t - 14 = 0 \text{ when } t = \frac{7}{3}.$$

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

$$80. (a) a(t) = \cos t$$

$$v(t) = \int a(t) \, dt = \int \cos t \, dt = \sin t + C_1 = \sin t \text{ (since } v_0 = 0)$$

$$f(t) = \int v(t) \, dt = \int \sin t \, dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$$

$$f(t) = -\cos t + 4$$

$$(b) v(t) = 0 = \sin t \text{ for } t = k\pi, k = 0, 1, 2, \dots$$

$$77. x(t) = t^3 - 6t^2 + 9t - 2 \quad 0 \leq t \leq 5$$

$$(a) v(t) = x'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$$

$$a(t) = v'(t) = 6t - 12 = 6(t-2)$$

$$(b) v(t) > 0 \text{ when } 0 < t < 1 \text{ or } 3 < t < 5.$$

$$(c) a(t) = 6(t-2) = 0 \text{ when } t = 2.$$

$$v(2) = 3(1)(-1) = -3$$

$$79. v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$$

$$x(t) = \int v(t) \, dt = 2t^{1/2} + C$$

$$x(1) = 4 = 2(1) + C \Rightarrow C = 2$$

$$x(t) = 2t^{1/2} + 2 \text{ position function}$$

$$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}} \text{ acceleration}$$

$$81. (a) \quad v(0) = 25 \text{ km/hr} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$$

$$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$$

$$a(t) = a \text{ (constant acceleration)}$$

$$v(t) = at + C$$

$$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$$

$$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$$

$$\frac{550}{36} = 13a$$

$$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$$

$$(b) \quad s(t) = a \frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$$

$$s(13) = \frac{275}{234} \frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$$

$$82. \quad v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$$

$$30 \text{ mph} = 44 \text{ ft/sec}$$

$$15 \text{ mph} = 22 \text{ ft/sec}$$

$$a(t) = -a$$

$$v(t) = -at + 66$$

$$s(t) = -\frac{a}{2}t^2 + 66t \quad (\text{Let } s(0) = 0.)$$

$$v(t) = 0 \text{ after car moves 132 ft.}$$

$$-at + 66 = 0 \text{ when } t = \frac{66}{a}.$$

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$$a(t) = -16.5$$

$$v(t) = -16.5t + 66$$

$$s(t) = -8.25t^2 + 66t$$

$$(a) \quad -16.5t + 66 = 44$$

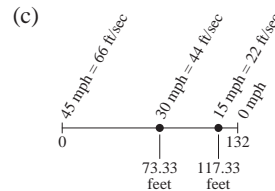
$$t = \frac{22}{16.5} \approx 1.333$$

$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

$$(b) \quad -16.5t + 66 = 22$$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$



It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

$$83. \text{ Truck: } v(t) = 30$$

$$s(t) = 30t \text{ (Let } s(0) = 0.)$$

$$\text{Automobile: } a(t) = 6$$

$$v(t) = 6t \text{ (Let } v(0) = 0.)$$

$$s(t) = 3t^2 \text{ (Let } s(0) = 0.)$$

At the point where the automobile overtakes the truck:

$$30t = 3t^2$$

$$0 = 3t^2 - 30t$$

$$0 = 3t(t - 10) \text{ when } t = 10 \text{ sec.}$$

$$(a) \quad s(10) = 3(10)^2 = 300 \text{ ft}$$

$$(b) \quad v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mph}$$

$$84. \frac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \frac{22}{15} \text{ ft/sec}$$

(a)

t	0	5	10	15	20	25	30
$V_1(\text{ft/sec})$	0	3.67	10.27	23.47	42.53	66	95.33
$V_2(\text{ft/sec})$	0	30.8	55.73	74.8	88	93.87	95.33

$$(b) V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$$

$$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$$

$$(c) S_1(t) = \int V_1(t) dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$$

$$S_2(t) = \int V_2(t) dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$.]

$$S_1(30) \approx 953.5 \text{ feet}$$

$$S_2(30) \approx 1970.3 \text{ feet}$$

The second car was going faster than the first until the end.

$$85. a(t) = k$$

$$v(t) = kt$$

$$s(t) = \frac{k}{2}t^2 \text{ since } v(0) = s(0) = 0.$$

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18,285.714 \text{ mi/hr}^2$$

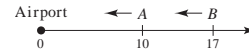
$$\approx 7.45 \text{ ft/sec}^2.$$

86. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.

$$v_A(t) = k_A t - 150$$

$$v_B = k_B t - 250$$

$$s_A(t) = \frac{1}{2}k_A t^2 - 150t + 10 \quad s_B = \frac{1}{2}k_B t^2 - 250t + 17$$



(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \Rightarrow k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}$$

—CONTINUED—

86. —CONTINUED—

$$k_A = \frac{50}{t_A} = 50 \left(\frac{125}{10} \right) = 625 \Rightarrow s_A(t) = \frac{625}{2} t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \Rightarrow k_B = \frac{135}{t_B}$$

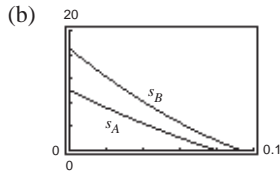
$$s_B(t_B) = \frac{1}{2} k_B t_B^2 - 250t_B + 17 = 0$$

$$\frac{1}{2} \left(\frac{135}{t_B} \right) t_B^2 - 250t_B = -17$$

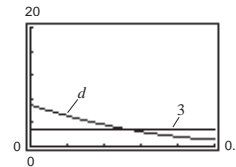
$$\frac{365}{2} t_B = 17$$

$$t_B = \frac{34}{365} \text{ hr.}$$

$$k_B = \frac{135}{t_B} = 135 \left(\frac{365}{34} \right) = \frac{49,275}{34} \Rightarrow s_B(t) = \frac{49,275}{68} t^2 - 250t + 17$$



(c) $d = s_B(t) - s_A(t) = \frac{28,025}{68} t^2 - 100t + 7$

Yes, $d < 3$ for $t > 0.0505$ hr.

87. True

88. True

89. True

90. True

91. False. For example, $\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1 \right) \left(\frac{x^2}{2} + C_2 \right)$.

92. False. f has an infinite number of antiderivatives, each differing by a constant.

93. $f''(x) = 2x$

$$f'(x) = x^2 + C$$

$$f'(2) = 0 \Rightarrow 4 + C = 0 \Rightarrow C = -4$$

$$f(x) = \frac{x^3}{3} - 4x + C_1$$

$$f(2) = 0 \Rightarrow \frac{8}{3} - 8 + C_1 = 0 \Rightarrow C_1 = \frac{16}{3}$$

$$\text{Answer: } f(x) = \frac{x^3}{3} - 4x + \frac{16}{3}$$

$$94. f'(x) = \begin{cases} -1, & 0 \leq x < 2 \\ 2, & 2 < x < 3 \\ 0, & 3 < x \leq 4 \end{cases}$$

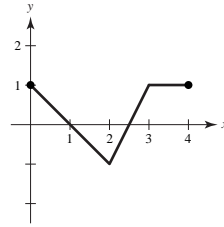
$$f(x) = \begin{cases} -x + C_1, & 0 \leq x < 2 \\ 2x + C_2, & 2 < x < 3 \\ C_3, & 3 < x \leq 4 \end{cases}$$

$$f(0) = 1 \Rightarrow C_1 = 1$$

$$f \text{ continuous at } x = 2 \Rightarrow -2 + 1 = 4 + C_2 \Rightarrow C_2 = -5$$

$$f \text{ continuous at } x = 3 \Rightarrow 6 - 5 = C_3 = 1$$

$$f(x) = \begin{cases} -x + 1, & 0 \leq x < 2 \\ 2x - 5, & 2 \leq x < 3 \\ 1, & 3 \leq x \leq 4 \end{cases}$$



$$95. f'(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 3x, & 2 \leq x \leq 5 \end{cases}$$

$$f(x) = \begin{cases} x + C_1, & 0 \leq x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \leq x \leq 5 \end{cases}$$

$$f(1) = 3 \Rightarrow 1 + C_1 = 3 \Rightarrow C_1 = 2$$

f is continuous: Values must agree at $x = 2$:

$$4 = 6 + C_2 \Rightarrow C_2 = -2$$

$$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ \frac{3x^2}{2} - 2, & 2 \leq x \leq 5 \end{cases}$$

The left and right hand derivatives at $x = 2$ do not agree. Hence f is not differentiable at $x = 2$.

$$96. \frac{d}{dx} [s(x)^2 + c(x)^2] = 2s(x)s'(x) + 2c(x)c'(x) \\ = 2s(x)c(x) - 2c(x)s(x) \\ = 0$$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k . Since, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore, $[s(x)]^2 + [c(x)]^2 = 1$.

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

$$97. f(x + y) = f(x)f(y) - g(x)g(y)$$

$$g(x + y) = f(x)g(y) + g(x)f(y)$$

$$f'(0) = 0$$

[Note: $f(x) = \cos x$ and $g(x) = \sin x$ satisfy these conditions]

$$f'(x + y) = f(x)f'(y) - g(x)g'(y) \text{ (Differentiate with respect to } y)$$

$$g'(x + y) = f(x)g'(y) + g(x)f'(y) \text{ (Differentiate with respect to } y)$$

$$\text{Letting } y = 0, f'(x) = f(x)f'(0) - g(x)g'(0) = -g(x)g'(0)$$

$$g'(x) = f(x)g'(0) + g(x)f'(0) = f(x)g'(0)$$

—CONTINUED—

97. —CONTINUED—

$$\text{Hence, } 2f(x)f'(x) = -2f(x)g(x)g'(x)$$

$$2g(x)g'(x) = 2g(x)f(x)g'(x)$$

$$\text{Adding, } 2f(x)f'(x) + 2g(x)g'(x) = 0.$$

$$\text{Integrating, } f(x)^2 + g(x)^2 = C.$$

Clearly $C \neq 0$, for if $C = 0$, then $f(x)^2 = -g(x)^2 \Rightarrow f(x) = g(x) = 0$, which contradicts that f, g are nonconstant.

$$\begin{aligned} \text{Now, } C &= f(x+y)^2 + g(x+y)^2 = (f(x)f(y) - g(x)g(y))^2 + (f(x)g(y) + g(x)f(y))^2 \\ &= f(x)^2f(y)^2 + g(x)^2g(y)^2 + f(x)^2g(y)^2 + g(x)^2f(y)^2 \\ &= [f(x)^2 + g(x)^2][f(y)^2 + g(y)^2] \\ &= C^2 \end{aligned}$$

Thus, $C = 1$ and we have $f(x)^2 + g(x)^2 = 1$.

Section 4.2 Area

$$1. \sum_{i=1}^5 (2i+1) = 2\sum_{i=1}^5 i + \sum_{i=1}^5 1 = 2(1+2+3+4+5) + 5 = 35$$

$$2. \sum_{k=3}^6 k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$$

$$3. \sum_{k=0}^4 \frac{1}{k^2+1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$$

$$4. \sum_{j=3}^5 \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

$$5. \sum_{k=1}^4 c = c + c + c + c = 4c$$

$$6. \sum_{i=1}^4 [(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$$

$$7. \sum_{i=1}^9 \frac{1}{3i}$$

$$8. \sum_{i=1}^{15} \frac{5}{1+i}$$

$$9. \sum_{j=1}^8 \left[5\left(\frac{j}{8}\right) + 3 \right]$$

$$10. \sum_{j=1}^4 \left[1 - \left(\frac{j}{4}\right)^2 \right]$$

$$11. \frac{2}{n} \sum_{i=1}^n \left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right) \right]$$

$$12. \frac{2}{n} \sum_{i=1}^n \left[1 - \left(\frac{2i}{n} - 1\right)^2 \right]$$

$$13. \frac{3}{n} \sum_{i=1}^n \left[2\left(1 + \frac{3i}{n}\right)^2 \right]$$

$$14. \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{1 - \left(\frac{i}{n}\right)^2}$$

$$15. \sum_{i=1}^{20} 2i = 2\sum_{i=1}^{20} i = 2\left[\frac{20(21)}{2}\right] = 420$$

$$\begin{aligned} 16. \sum_{i=1}^{15} (2i-3) &= 2\sum_{i=1}^{15} i - 3(15) \\ &= 2\left[\frac{15(16)}{2}\right] - 45 = 195 \end{aligned}$$

$$\begin{aligned} 17. \sum_{i=1}^{20} (i-1)^2 &= \sum_{i=1}^{19} i^2 \\ &= \left[\frac{19(20)(39)}{6}\right] = 2470 \end{aligned}$$

$$\begin{aligned} 18. \sum_{i=1}^{10} (i^2-1) &= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 \\ &= \left[\frac{10(11)(21)}{6}\right] - 10 = 375 \end{aligned}$$

$$\begin{aligned}
 19. \sum_{i=1}^{15} i(i-1)^2 &= \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i \\
 &= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2} \\
 &= 14,400 - 2480 + 120 \\
 &= 12,040
 \end{aligned}$$

$$\begin{aligned}
 20. \sum_{i=1}^{10} i(i^2 + 1) &= \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i \\
 &= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2} \right] = 3080
 \end{aligned}$$

$$21. \text{sum seq}(x \square 2 + 3, x, 1, 20, 1) = 2930 \quad (TI-82)$$

$$\begin{aligned}
 \sum_{i=1}^{20} (i^2 + 3) &= \frac{20(20+1)(2(20)+1)}{6} + 3(20) \\
 &= \frac{(20)(21)(41)}{6} + 60 = 2930
 \end{aligned}$$

$$22. \text{sum seq}(x \square 3 - 2x, x, 1, 15, 1) = 14,160 \quad (TI-82)$$

$$\begin{aligned}
 \sum_{i=1}^{15} (i^3 - 2i) &= \frac{(15)^2(15+1)^2}{4} - 2\frac{15(15+1)}{2} \\
 &= \frac{(15)^2(16)^2}{4} - 15(16) = 14,160
 \end{aligned}$$

$$\begin{aligned}
 23. S &= [3 + 4 + \frac{9}{2} + 5](1) = \frac{33}{2} = 16.5 \\
 s &= [1 + 3 + 4 + \frac{9}{2}](1) = \frac{25}{2} = 12.5
 \end{aligned}$$

$$\begin{aligned}
 24. S &= [5 + 5 + 4 + 2](1) = 16 \\
 s &= [4 + 4 + 2 + 0](1) = 10
 \end{aligned}$$

$$\begin{aligned}
 25. S &= [3 + 3 + 5](1) = 11 \\
 s &= [2 + 2 + 3](1) = 7
 \end{aligned}$$

$$\begin{aligned}
 26. S &= \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2} \right] = \frac{55}{6} \\
 s &= \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3} \right] = \frac{9}{2} = 4.5
 \end{aligned}$$

$$\begin{aligned}
 27. S(4) &= \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768 \\
 s(4) &= 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518
 \end{aligned}$$

$$\begin{aligned}
 28. S(8) &= \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2 \right) \frac{1}{4} + (\sqrt{1} + 2) \frac{1}{4} \\
 &\quad + \left(\sqrt{\frac{5}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} + (\sqrt{2} + 2) \frac{1}{4} \\
 &= \frac{1}{4} \left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2} \right) \approx 6.038 \\
 s(8) &= (0 + 2) \frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} \approx 5.685
 \end{aligned}$$

$$\begin{aligned}
 29. S(5) &= 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746 \\
 s(5) &= \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646
 \end{aligned}$$

$$\begin{aligned}
 30. S(5) &= 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) \\
 &= \frac{1}{5} \left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5} \right] \approx 0.859 \\
 s(5) &= \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659
 \end{aligned}$$

$$31. \lim_{n \rightarrow \infty} \left[\left(\frac{81}{n^4} \right) \frac{n^2(n+1)^2}{4} \right] = \frac{81}{4} \lim_{n \rightarrow \infty} \left[\frac{n^4 + 2n^3 + n^2}{n^4} \right] = \frac{81}{4}(1) = \frac{81}{4}$$

$$32. \lim_{n \rightarrow \infty} \left[\left(\frac{64}{n^3} \right) \frac{n(n+1)(2n+1)}{6} \right] = \frac{64}{6} \lim_{n \rightarrow \infty} \left[\frac{2n^3 + 3n^2 + n}{n^3} \right] = \frac{64}{6}(2) = \frac{64}{3}$$

$$33. \lim_{n \rightarrow \infty} \left[\left(\frac{18}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{18}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{18}{2}(1) = 9 \quad 34. \lim_{n \rightarrow \infty} \left[\left(\frac{1}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{1}{2}(1) = \frac{1}{2}$$

$$35. \sum_{i=1}^n \frac{2i+1}{n^2} = \frac{1}{n^2} \sum_{i=1}^n (2i+1) = \frac{1}{n^2} \left[2 \frac{n(n+1)}{2} + n \right] = \frac{n+2}{n} = 1 + \frac{2}{n} = S(n)$$

$$S(10) = \frac{12}{10} = 1.2$$

$$S(100) = 1.02$$

$$S(1000) = 1.002$$

$$S(10,000) = 1.0002$$

$$36. \sum_{j=1}^n \frac{4j+3}{n^2} = \frac{1}{n^2} \sum_{j=1}^n (4j+3) = \frac{1}{n^2} \left[\frac{4n(n+1)}{2} + 3n \right] = \frac{2n+5}{n} = S(n)$$

$$S(10) = \frac{25}{10} = 2.5$$

$$S(100) = 2.05$$

$$S(1000) = 2.005$$

$$S(10,000) = 2.0005$$

$$37. \sum_{k=1}^n \frac{6k(k-1)}{n^3} = \frac{6}{n^3} \sum_{k=1}^n (k^2 - k) = \frac{6}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right]$$

$$= \frac{6}{n^3} \left[\frac{2n^2 + 3n + 1 - 3n - 3}{6} \right] = \frac{1}{n^2} [2n^2 - 2] = 2 - \frac{2}{n^2} = S(n)$$

$$S(10) = 1.98$$

$$S(100) = 1.9998$$

$$S(1000) = 1.999998$$

$$S(10,000) = 1.99999998$$

$$38. \sum_{i=1}^n \frac{4i^2(i-1)}{n^4} = \frac{4}{n^4} \sum_{i=1}^n (i^3 - i^2) = \frac{4}{n^4} \left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{4}{n^3} \left[\frac{n^3 + 2n^2 + n}{4} - \frac{2n^2 + 3n + 1}{6} \right]$$

$$= \frac{1}{3n^3} [3n^3 + 6n^2 + 3n - 4n^2 - 6n - 2]$$

$$= \frac{1}{3n^3} [3n^3 + 2n^2 - 3n - 2] = S(n)$$

$$S(10) = 1.056$$

$$S(100) = 1.006566$$

$$S(1000) = 1.00066567$$

$$S(10,000) = 1.000066657$$

$$39. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{16i}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{16}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{16}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \left[8 \left(\frac{n^2+n}{n^2} \right) \right] = 8 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 8$$

$$40. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n} \right) \left(\frac{2}{n} \right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{4}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \frac{4}{2} \left(1 + \frac{1}{n} \right) = 2$$

$$41. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i-1)^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{(n-1)(n)(2n-1)}{6} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{6} \left[\frac{2n^3 - 3n^2 + n}{n^3} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(\frac{2 - (3/n) + (1/n^2)}{1} \right) \right] = \frac{1}{3}$$

$$42. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n} \right)^2 \left(\frac{2}{n} \right) = \lim_{n \rightarrow \infty} \frac{2}{n^3} \sum_{i=1}^n (n+2i)^2$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[\sum_{i=1}^n n^2 + 4n \sum_{i=1}^n i + 4 \sum_{i=1}^n i^2 \right]$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[n^3 + (4n) \left(\frac{n(n+1)}{2} \right) + \frac{4(n)(n+1)(2n+1)}{6} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2} \right]$$

$$= 2 \left(1 + 2 + \frac{4}{3} \right) = \frac{26}{3}$$

$$43. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{i}{n} \right) \left(\frac{2}{n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{i=1}^n 1 + \frac{1}{n} \sum_{i=1}^n i \right] = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{1}{n} \left(\frac{n(n+1)}{2} \right) \right] = 2 \lim_{n \rightarrow \infty} \left[1 + \frac{n^2+n}{2n^2} \right] = 2 \left(1 + \frac{1}{2} \right) = 3$$

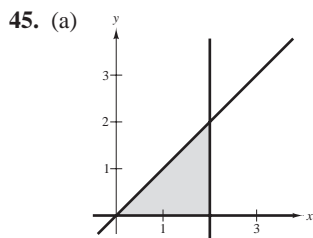
$$44. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n} \right)^3 \left(\frac{2}{n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n+2i)^3$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n^3 + 6n^2i + 12ni^2 + 8i^3)$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[n^4 + 6n^2 \left(\frac{n(n+1)}{2} \right) + 12n \left(\frac{n(n+1)(2n+1)}{6} \right) + 8 \left(\frac{n^2(n+1)^2}{4} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left(1 + 3 + \frac{3}{n} + 4 + \frac{6}{n} + \frac{2}{n^2} + 2 + \frac{4}{n} + \frac{2}{n^2} \right)$$

$$= 2 \lim_{n \rightarrow \infty} \left(10 + \frac{13}{n} + \frac{4}{n^2} \right) = 20$$



$$(b) \Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Endpoints:

$$0 < 1 \left(\frac{2}{n} \right) < 2 \left(\frac{2}{n} \right) < \dots < (n-1) \left(\frac{2}{n} \right) < n \left(\frac{2}{n} \right) = 2$$

45. —CONTINUED—

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^n f(x_{i-1}) \Delta x$$

$$= \sum_{i=1}^n f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

$$(f) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[(i-1)\left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n (i-1)$$

$$= \lim_{n \rightarrow \infty} \frac{4}{n^2} \left[\frac{n(n+1)}{2} - n \right]$$

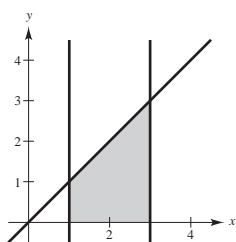
$$= \lim_{n \rightarrow \infty} \left[\frac{2(n+1)}{n} - \frac{4}{n} \right] = 2$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[i\left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i$$

$$= \lim_{n \rightarrow \infty} \left(\frac{4}{n^2} \right) \frac{n(n+1)}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 2$$

46. (a)



$$(b) \Delta x = \frac{3-1}{n} = \frac{2}{n}$$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \dots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \dots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^n f(x_{i-1}) \Delta x$$

$$= \sum_{i=1}^n f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

$$(f) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \left(\frac{2}{n}\right) \left[n + \frac{2}{n} \left(\frac{n(n+1)}{2} - n \right) \right]$$

$$= \lim_{n \rightarrow \infty} \left[2 + \frac{2n+2}{n} - \frac{4}{n} \right] = \lim_{n \rightarrow \infty} \left[4 - \frac{2}{n} \right] = 4$$

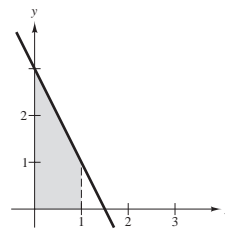
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{2}{n} \left[n + \left(\frac{2}{n}\right) \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \rightarrow \infty} \left[2 + \frac{2(n+1)}{n} \right] = \lim_{n \rightarrow \infty} \left[4 + \frac{2}{n} \right] = 4$$

47. $y = -2x + 3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[-2\left(\frac{i}{n}\right) + 3\right]\left(\frac{1}{n}\right) \\ &= 3 - \frac{2}{n^2} \sum_{i=1}^n i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n} \end{aligned}$$

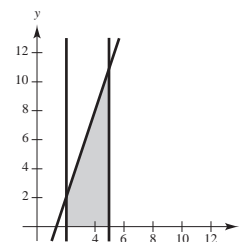
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 2$$



48. $y = 3x - 4$ on $[2, 5]$. (Note: $\Delta x = \frac{5-2}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \sum_{i=1}^n \left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^n i - 12 \\ &= 6 + \frac{27}{n^2} \left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2} \left(1 + \frac{1}{n}\right) \end{aligned}$$

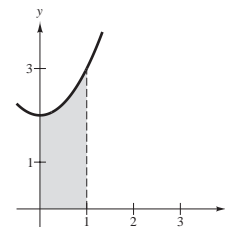
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$



49. $y = x^2 + 2$ on $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right) \\ &= \left[\frac{1}{n^3} \sum_{i=1}^n i^2\right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2 \end{aligned}$$

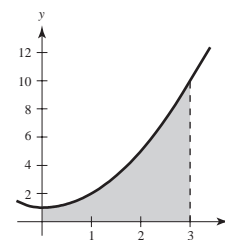
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \frac{7}{3}$$



50. $y = x^2 + 1$ on $[0, 3]$. (Note: $\Delta x = \frac{3-0}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n^3} \sum_{i=1}^n i^2 + \frac{3}{n} \sum_{i=1}^n 1 \\ &= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2} \frac{2n^2 + 3n + 1}{n^2} + 3 \end{aligned}$$

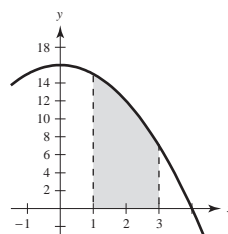
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \frac{9}{2}(2) + 3 = 12$$



51. $y = 16 - x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{2}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n} \sum_{i=1}^n \left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right] \\ &= \frac{2}{n} \left[15n - \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n} \frac{n(n+1)}{2}\right] \\ &= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1) \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$

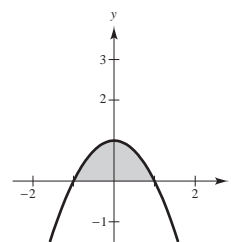


52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\ &= 1 - \frac{1}{n^3} \sum_{i=1}^n i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\frac{1}{2} \text{Area} = \lim_{n \rightarrow \infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

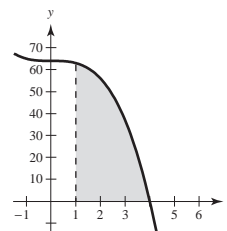
$$\text{Area} = \frac{4}{3}$$



53. $y = 64 - x^3$ on $[1, 4]$. (Note: $\Delta x = \frac{4-1}{n} = \frac{3}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right] \\ &= \frac{3}{n} \left[63n - \frac{27}{n^3} \frac{n^2(n+1)^2}{4} - \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{9}{n} \frac{n(n+1)}{2}\right] \\ &= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2} \frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

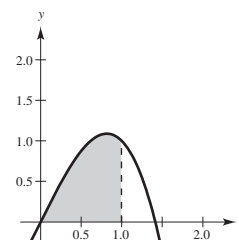


54. $y = 2x - x^3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) \\ &= \frac{2}{n^2} \sum_{i=1}^n i - \frac{1}{n^4} \sum_{i=1}^n i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right] \\ &= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

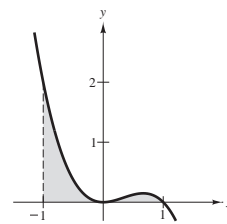


55. $y = x^2 - x^3$ on $[-1, 1]$. (Note: $\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}$)

Again, $T(n)$ is neither an upper nor a lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3 \right] \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right) \right] \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3} \right] \left(\frac{2}{n}\right) = \frac{4}{n} \sum_{i=1}^n 1 - \frac{20}{n^2} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n i^2 - \frac{16}{n^4} \sum_{i=1}^n i^3 \\ &= \frac{4}{n}(n) - \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \end{aligned}$$

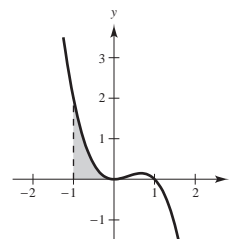
$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$



56. $y = x^2 - x^3$ on $[-1, 0]$. (Note: $\Delta x = \frac{0 - (-1)}{n} = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3 \right] \left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right) \right] \left(\frac{1}{n}\right) = 2 - \frac{5}{n^2} \sum_{i=1}^n i + \frac{4}{n^3} \sum_{i=1}^n i^2 - \frac{1}{n^4} \sum_{i=1}^n i^3 \\ &= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} \end{aligned}$$

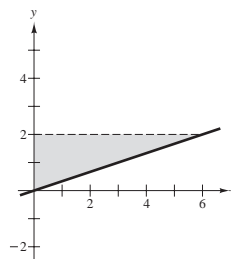
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$



57. $f(y) = 3y$, $0 \leq y \leq 2$ (Note: $\Delta y = \frac{2 - 0}{n} = \frac{2}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f(m_i) \Delta y = \sum_{i=1}^n f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \frac{12}{n^2} \sum_{i=1}^n i = \left(\frac{12}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n} \end{aligned}$$

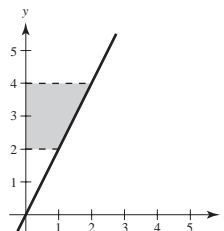
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(6 + \frac{6}{n}\right) = 6$$



58. $g(y) = \frac{1}{2}y$, $2 \leq y \leq 4$. (Note: $\Delta y = \frac{4 - 2}{n} = \frac{2}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \\ &= \frac{2}{n} \left[n + \frac{1}{n} \frac{n(n+1)}{2} \right] = 2 + \frac{n+1}{n} \end{aligned}$$

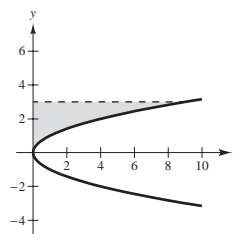
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + 1 = 3$$



59. $f(y) = y^2, 0 \leq y \leq 3$ (Note: $\Delta y = \frac{3-0}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \frac{27}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2} \left(\frac{2n^2 + 3n + 1}{2} \right) = 9 + \frac{27}{2n} + \frac{9}{2n^2} \end{aligned}$$

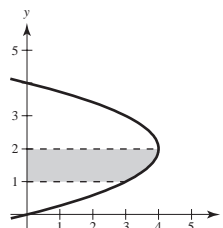
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(9 + \frac{27}{2n} + \frac{9}{2n^2} \right) = 9$$



60. $f(y) = 4y - y^2, 1 \leq y \leq 2$. (Note: $\Delta y = \frac{2-1}{n} = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n \left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(3 + \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \left[3n + \frac{2n(n+1)}{2} - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6} \end{aligned}$$

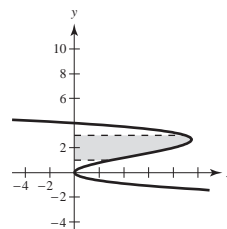
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



61. $g(y) = 4y^2 - y^3, 1 \leq y \leq 3$. (Note: $\Delta y = \frac{3-1}{n} = \frac{2}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3 \right] \frac{2}{n} \\ &= \frac{2}{n} \sum_{i=1}^n 4 \left[1 + \frac{4i}{n} + \frac{4i^2}{n^2} \right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3} \right] \\ &= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3} \right] = \frac{2}{n} \left[3n + \frac{10n(n+1)}{2} + \frac{4n(n+1)(2n+1)}{6} - \frac{8n^2(n+1)^2}{4} \right] \end{aligned}$$

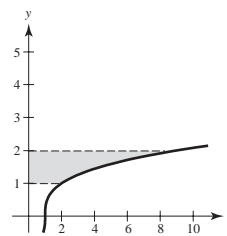
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



62. $h(y) = y^3 + 1, 1 \leq y \leq 2$ (Note: $\Delta y = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^3 + 1 \right] \frac{1}{n} \\ &= \frac{1}{n} \sum_{i=1}^n \left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n} \right) \\ &= \frac{1}{n} \left[2n + \frac{1}{n^3} \frac{n^2(n+1)^2}{4} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \frac{n(n+1)}{2} \right] \\ &= 2 + \frac{(n+1)^2}{n^2} + \frac{1}{2} \frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



63. $f(x) = x^2 + 3, 0 \leq x \leq 2, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 3] \left(\frac{1}{2}\right) \\ &= \frac{1}{2} \left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right) \right] \\ &= \frac{69}{8} \end{aligned}$$

65. $f(x) = \tan x, 0 \leq x \leq \frac{\pi}{4}, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\tan c_i) \left(\frac{\pi}{16}\right) \\ &= \frac{\pi}{16} \left(\tan \frac{\pi}{32} + \tan \frac{3\pi}{32} + \tan \frac{5\pi}{32} + \tan \frac{7\pi}{32} \right) \approx 0.345 \end{aligned}$$

67. $f(x) = \sqrt{x}$ on $[0, 4]$.

n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is $16/3$.)

64. $f(x) = x^2 + 4x, 0 \leq x \leq 4, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = 1, c_1 = \frac{1}{2}, c_2 = \frac{3}{2}, c_3 = \frac{5}{2}, c_4 = \frac{7}{2}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 4c_i](1) \\ &= \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right) \right] \\ &= 53 \end{aligned}$$

66. $f(x) = \sin x, 0 \leq x \leq \frac{\pi}{2}, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{\pi}{8}, c_1 = \frac{\pi}{16}, c_2 = \frac{3\pi}{16}, c_3 = \frac{5\pi}{16}, c_4 = \frac{7\pi}{16}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\sin c_i) \left(\frac{\pi}{8}\right) \\ &= \frac{\pi}{8} \left(\sin \frac{\pi}{16} + \sin \frac{3\pi}{16} + \sin \frac{5\pi}{16} + \sin \frac{7\pi}{16} \right) \approx 1.006 \end{aligned}$$

68. $f(x) = \frac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

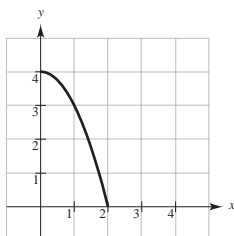
69. $f(x) = \tan\left(\frac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

70. $f(x) = \cos\sqrt{x}$ on $[0, 2]$.

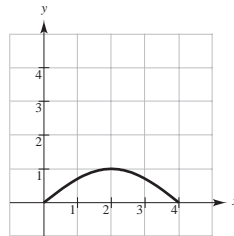
n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

71.



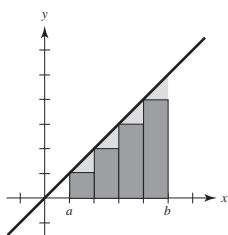
(b) $A \approx 6$ square units

72.

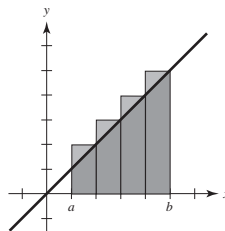


(a) $A \approx 3$ square units

73. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



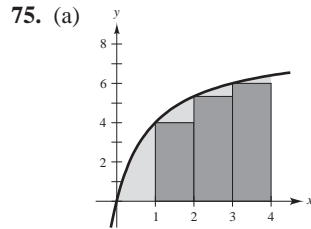
The sum of the areas of these circumscribed rectangles is the upper sum.



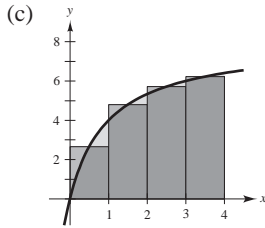
We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

74. See the definition of area. Page 265.



Lower sum:
 $s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$



Midpoint Rule:
 $M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

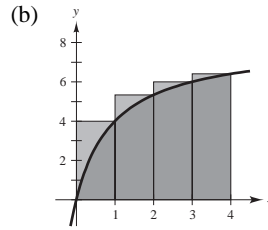
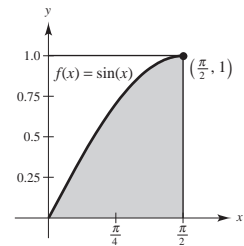
(f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

76. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$

Let A_1 = area bounded by $f(x) = \sin x$, the x -axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1, y = 0, x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$. In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2 . It is keeping track of how many of these points, N_1 , lie in the region whose area is represented by A_1 . Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

The larger N_2 is the better the approximation to A_1 .



Upper sum:
 $S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{9} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $(i - \frac{1}{2})(4/n)$.

77. True. (Theorem 4.2 (2))

78. True. (Theorem 4.3)

79. Suppose there are n rows and $n + 1$ columns in the figure. The stars on the left total $1 + 2 + \dots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

$$2[1 + 2 + \dots + n] = n(n + 1)$$

$$1 + 2 + \dots + n = \frac{1}{2}n(n + 1).$$

80. (a) $\theta = \frac{2\pi}{n}$

(b) $\sin \theta = \frac{h}{r}$

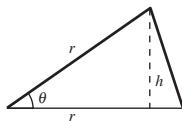
$$h = r \sin \theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2 \sin \theta$$

(c) $A_n = n \left(\frac{1}{2}r^2 \sin \frac{2\pi}{n} \right) = \frac{r^2 n}{2} \sin \frac{2\pi}{n} = \pi r^2 \left(\frac{\sin(2\pi/n)}{2\pi/n} \right)$

Let $x = 2\pi/n$. As $n \rightarrow \infty$, $x \rightarrow 0$.

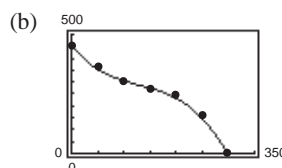
$$\lim_{n \rightarrow \infty} A_n = \lim_{x \rightarrow 0} \pi r^2 \left(\frac{\sin x}{x} \right) = \pi r^2(1) = \pi r^2$$



81. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76,897.5 \text{ ft}^2$$



82. For n odd,

$$\begin{aligned} n = 1, & \quad 1 \text{ row}, & \quad 1 \text{ block} \\ n = 3, & \quad 2 \text{ rows}, & \quad 4 \text{ blocks} \\ n = 5, & \quad 3 \text{ rows}, & \quad 9 \text{ blocks} \\ n, & \quad \frac{n+1}{2} \text{ rows}, & \quad \left(\frac{n+1}{2} \right)^2 \text{ blocks,} \end{aligned}$$

For n even,

$$\begin{aligned} n = 2, & \quad 1 \text{ row}, & \quad 2 \text{ block} \\ n = 4, & \quad 2 \text{ rows}, & \quad 6 \text{ blocks} \\ n = 6, & \quad 3 \text{ rows}, & \quad 12 \text{ blocks} \\ n, & \quad \frac{n}{2} \text{ rows}, & \quad \frac{n^2 + 2n}{4} \text{ blocks,} \end{aligned}$$

83. (a) $\sum_{i=1}^n 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$.

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k 2i = k(k+1).$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} 2i &= \sum_{i=1}^k 2i + 2(k+1) \\ &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

which shows that the formula is true for $n = k+1$.

(b) $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1.$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}.$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4} [k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4} (k+2)^2 \end{aligned}$$

which shows that the formula is true for $n = k+1$.

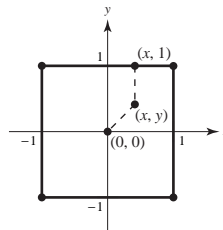
84. Assume that the dartboard has corners at $(\pm 1, \pm 1)$.

A point (x, y) in the square is closer to the center than the top edge if

$$\sqrt{x^2 + y^2} \leq 1 - y$$

$$x^2 + y^2 \leq 1 - 2y + y^2$$

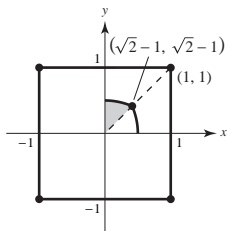
$$y \leq \frac{1}{2}(1 - x^2).$$



By symmetry, a point (x, y) in the square is closer to the center than the right edge if

$$x \leq \frac{1}{2}(1 - y^2).$$

In the first quadrant, the parabolas $y = \frac{1}{2}(1 - x^2)$ and $x = \frac{1}{2}(1 - y^2)$ intersect at $(\sqrt{2} - 1, \sqrt{2} - 1)$. There are 8 equal regions that make up the total region, as indicated in the figure.



$$\text{Area of shaded region } S = \int_0^{\sqrt{2}-1} \left[\frac{1}{2}(1 - x^2) - x \right] dx = \frac{2\sqrt{2}}{3} - \frac{5}{6}$$

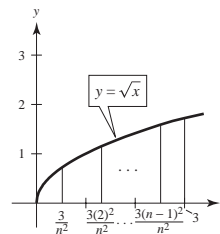
$$\text{Probability} = \frac{8S}{\text{Area square}} = 2 \left[\frac{2\sqrt{2}}{3} - \frac{5}{6} \right] = \frac{4\sqrt{2}}{3} - \frac{5}{3}$$

Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x}$, $y = 0$, $x = 0$, $x = 3$, $c_i = \frac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i-1)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\frac{3i^2}{n^2}} \frac{3}{n^2}(2i-1) \\ &= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^n (2i^2 - i) \\ &= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \left[2 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} 3\sqrt{3} \left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} \right] \\ &= 3\sqrt{3} \left[\frac{2}{3} - 0 \right] = 2\sqrt{3} \approx 3.464 \end{aligned}$$



$$2. f(x) = \sqrt[3]{x}, y = 0, x = 0, x = 1, c_i = \frac{i^3}{n^3}$$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt[3]{\frac{i^3}{n^3}} \left[\frac{3i^2 - 3i + 1}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (3i^3 - 3i^2 + i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[3 \left(\frac{n^2(n+1)^2}{4} \right) - 3 \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4} \right] = \lim_{n \rightarrow \infty} \left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2} \right] = \frac{3}{4} \end{aligned}$$

$$3. y = 6 \text{ on } [4, 10]. \quad \left(\text{Note: } \Delta x = \frac{10-4}{n} = \frac{6}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty \right)$$

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(4 + \frac{6i}{n}\right) \left(\frac{6}{n}\right) = \sum_{i=1}^n 6 \left(\frac{6}{n}\right) = \sum_{i=1}^n \frac{36}{n} = \frac{1}{n} \sum_{i=1}^n 36 = \frac{1}{n} (36n) = 36 \\ \int_4^{10} 6 \, dx &= \lim_{n \rightarrow \infty} 36 = 36 \end{aligned}$$

$$4. y = x \text{ on } [-2, 3]. \quad \left(\text{Note: } \Delta x = \frac{3 - (-2)}{n} = \frac{5}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty \right)$$

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = \sum_{i=1}^n \left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = -10 + \frac{25}{n^2} \sum_{i=1}^n i \\ &= -10 + \left(\frac{25}{n^2}\right) \frac{n(n+1)}{2} = -10 + \frac{25}{2} \left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n} \\ \int_{-2}^3 x \, dx &= \lim_{n \rightarrow \infty} \left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2} \end{aligned}$$

$$5. y = x^3 \text{ on } [-1, 1]. \quad \left(\text{Note: } \Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty \right)$$

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \left(\frac{2}{n}\right) \\ &= -2 + \frac{12}{n^2} \sum_{i=1}^n i - \frac{24}{n^3} \sum_{i=1}^n i^2 + \frac{16}{n^4} \sum_{i=1}^n i^3 \\ &= -2 + 6 \left(1 + \frac{1}{n}\right) - 4 \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4 \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n} \\ \int_{-1}^1 x^3 \, dx &= \lim_{n \rightarrow \infty} \frac{2}{n} = 0 \end{aligned}$$

6. $y = 3x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{3-1}{n} = \frac{2}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n 3 \left(1 + \frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) \\ &= \frac{6}{n} \sum_{i=1}^n \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) \\ &= \frac{6}{n} \left[n + \frac{4}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 6 + 12 \frac{n+1}{n} + 4 \frac{(n+1)(2n+1)}{n^2} \\ \int_1^3 3x^2 dx &= \lim_{n \rightarrow \infty} \left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2} \right] \\ &= 6 + 12 + 8 = 26 \end{aligned}$$

7. $y = x^2 + 1$ on $[1, 2]$. (Note: $\Delta x = \frac{2-1}{n} = \frac{1}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(1 + \frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^2 + 1 \right] \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1 \right] \left(\frac{1}{n}\right) \\ &= 2 + \frac{2}{n^2} \sum_{i=1}^n i + \frac{1}{n^3} \sum_{i=1}^n i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2} \\ \int_1^2 (x^2 + 1) dx &= \lim_{n \rightarrow \infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2} \right) = \frac{10}{3} \end{aligned}$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. (Note: $\Delta x = \frac{2 - (-1)}{n} = \frac{3}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-1 + \frac{3i}{n}\right) \left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(-1 + \frac{3i}{n}\right)^2 + 2 \right] \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2 \right] \\ &= \frac{3}{n} \left[3n - \frac{18}{n} \frac{n(n+1)}{2} + \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} + 2n \right] \\ &= 15 - \frac{27(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2} \\ \int_{-1}^2 (3x^2 + 2) dx &= \lim_{n \rightarrow \infty} \left[15 - 27 \frac{(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= 15 - 27 + 27 = 15 \end{aligned}$$

9. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (3c_i + 10) \Delta x_i = \int_{-1}^5 (3x + 10) dx$
on the interval $[-1, 5]$.

10. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 6c_i(4 - c_i)^2 \Delta x_i = \int_0^4 6x(4 - x)^2 dx$
on the interval $[0, 4]$.

$$11. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt{c_i^2 + 4} \Delta x_i = \int_0^3 \sqrt{x^2 + 4} dx$$

on the interval $[0, 3]$.

$$12. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left(\frac{3}{c_i^2}\right) \Delta x_i = \int_1^3 \frac{3}{x^2} dx$$

on the interval $[1, 3]$.

$$13. \int_0^5 3 dx$$

$$14. \int_0^2 (4 - 2x) dx$$

$$15. \int_{-4}^4 (4 - |x|) dx$$

$$16. \int_0^2 x^2 dx$$

$$17. \int_{-2}^2 (4 - x^2) dx$$

$$18. \int_{-1}^1 \frac{1}{x^2 + 1} dx$$

$$19. \int_0^{\pi} \sin x dx$$

$$20. \int_0^{\pi/4} \tan x dx$$

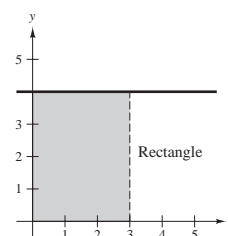
$$21. \int_0^2 y^3 dy$$

$$22. \int_0^2 (y - 2)^2 dy$$

23. Rectangle

$$A = bh = 3(4)$$

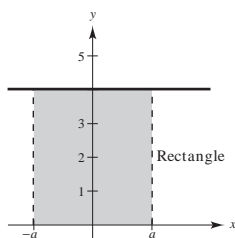
$$A = \int_0^3 4 dx = 12$$



24. Rectangle

$$A = bh = 2(4)(a)$$

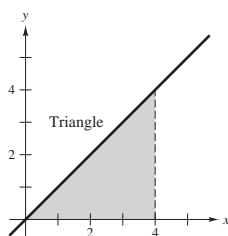
$$A = \int_{-a}^a 4 dx = 8a$$



25. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

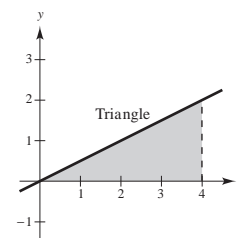
$$A = \int_0^4 x dx = 8$$



26. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$$

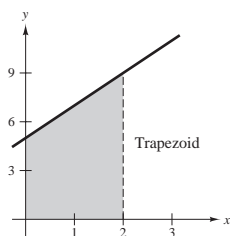
$$A = \int_0^4 \frac{x}{2} dx = 4$$



27. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$$

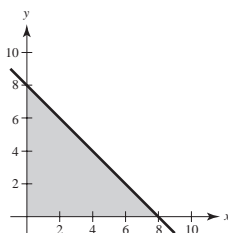
$$A = \int_0^2 (2x + 5) dx = 14$$



28. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$$

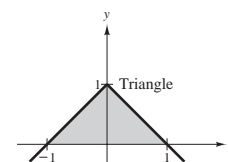
$$A = \int_0^8 (8 - x) dx = 32$$



29. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

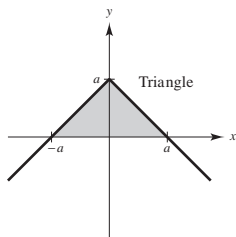
$$A = \int_{-1}^1 (1 - |x|) dx = 1$$



30. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$$

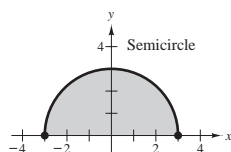
$$A = \int_{-a}^a (a - |x|) dx = a^2$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

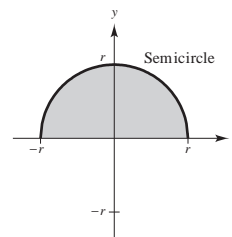
$$A = \int_{-3}^3 \sqrt{9 - x^2} dx = \frac{9\pi}{2}$$



32. Semicircle

$$A = \frac{1}{2}\pi r^2$$

$$A = \int_{-r}^r \sqrt{r^2 - x^2} dx = \frac{1}{2}\pi r^2$$



In Exercises 33–40, $\int_2^4 x^3 dx = 60$, $\int_2^4 x dx = 6$, $\int_2^4 dx = 2$

$$33. \int_4^2 x dx = -\int_2^4 x dx = -6$$

$$35. \int_2^4 4x dx = 4 \int_2^4 x dx = 4(6) = 24$$

$$37. \int_2^4 (x - 8) dx = \int_2^4 x dx - 8 \int_2^4 dx = 6 - 8(2) = -10$$

$$39. \int_2^4 \left(\frac{1}{2}x^3 - 3x + 2 \right) dx = \frac{1}{2} \int_2^4 x^3 dx - 3 \int_2^4 x dx + 2 \int_2^4 dx \\ = \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

$$41. (a) \int_0^7 f(x) dx = \int_0^5 f(x) dx + \int_5^7 f(x) dx = 10 + 3 = 13$$

$$(b) \int_5^0 f(x) dx = -\int_0^5 f(x) dx = -10$$

$$(c) \int_5^5 f(x) dx = 0$$

$$(d) \int_0^5 3f(x) dx = 3 \int_0^5 f(x) dx = 3(10) = 30$$

$$43. (a) \int_2^6 [f(x) + g(x)] dx = \int_2^6 f(x) dx + \int_2^6 g(x) dx \\ = 10 + (-2) = 8$$

$$(b) \int_2^6 [g(x) - f(x)] dx = \int_2^6 g(x) dx - \int_2^6 f(x) dx \\ = -2 - 10 = -12$$

$$(c) \int_2^6 2g(x) dx = 2 \int_2^6 g(x) dx = 2(-2) = -4$$

$$(d) \int_2^6 3f(x) dx = 3 \int_2^6 f(x) dx = 3(10) = 30$$

$$34. \int_2^2 x^3 dx = 0$$

$$36. \int_2^4 15 dx = 15 \int_2^4 dx = 15(2) = 30$$

$$38. \int_2^4 (x^3 + 4) dx = \int_2^4 x^3 dx + 4 \int_2^4 dx = 60 + 4(2) = 68$$

$$40. \int_2^4 (6 + 2x - x^3) dx = 6 \int_2^4 dx + 2 \int_2^4 x dx - \int_2^4 x^3 dx \\ = 6(2) + 2(6) - 60 = -36$$

$$42. (a) \int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 4 + (-1) = 3$$

$$(b) \int_6^3 f(x) dx = -\int_3^6 f(x) dx = -(-1) = 1$$

$$(c) \int_3^3 f(x) dx = 0$$

$$(d) \int_3^6 -5f(x) dx = -5 \int_3^6 f(x) dx = -5(-1) = 5$$

$$44. (a) \int_{-1}^0 f(x) dx = \int_{-1}^1 f(x) dx - \int_0^1 f(x) dx = 0 - 5 = -5$$

$$(b) \int_0^1 f(x) dx - \int_1^0 f(x) dx = 5 - (-5) = 10$$

$$(c) \int_{-1}^1 3f(x) dx = 3 \int_{-1}^1 f(x) dx = 3(0) = 0$$

$$(d) \int_0^1 3f(x) dx = 3 \int_0^1 f(x) dx = 3(5) = 15$$

45. Lower estimate: $[24 + 12 - 4 - 20 - 36](2) = -48$

Upper estimate: $[32 + 24 + 12 - 4 - 20](2) = 88$

46. (a) $[-6 + 8 + 30](2) = 64$ left endpoint estimate

(b) $[8 + 30 + 80](2) = 236$ right endpoint estimate

(c) $[0 + 18 + 50](2) = 136$ midpoint estimate

If f is increasing, then (a) is below the actual value and (b) is above.

47. (a) Quarter circle below x -axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x -axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

48. (a) $\int_0^1 -f(x) dx = -\int_0^1 f(x) dx = \frac{1}{2}$

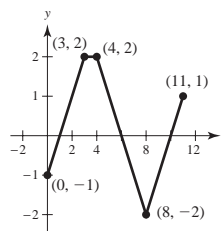
(b) $\int_3^4 3f(x) dx = 3(2) = 6$

(c) $\int_0^7 f(x) dx = -\frac{1}{2} + \frac{1}{2}(2)(2) + 2 + \frac{1}{2}(2)(2) - \frac{1}{2} = 5$

(d) $\int_5^{11} f(x) dx = \frac{1}{2} - \frac{1}{2}(4)(2) + \frac{1}{2} = -3$

(e) $\int_0^{11} f(x) dx = -\frac{1}{2} + 2 + 2 + 2 - 4 + \frac{1}{2} = 2$

(f) $\int_4^{10} f(x) dx = 2 - 4 = -2$



49. (a) $\int_0^5 [f(x) + 2] dx = \int_0^5 f(x) dx + \int_0^5 2 dx = 4 + 10 = 14$

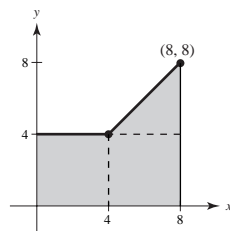
(b) $\int_{-2}^3 f(x+2) dx = \int_0^5 f(x) dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x) dx = 2 \int_0^5 f(x) dx = 2(4) = 8$ (f even)

(d) $\int_{-5}^5 f(x) dx = 0$ (f odd)

50. $f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$

$$\int_0^8 f(x) dx = 4(4) + 4(4) + \frac{1}{2}(4)(4) = 40$$


 51. The left endpoint approximation will be greater than the actual area: $>$

 52. The right endpoint approximation will be less than the actual area: $<$

53. $f(x) = \frac{1}{x-4}$

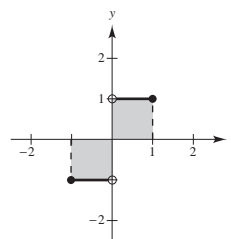
is not integrable on the interval $[3, 5]$ because f has a discontinuity at $x = 4$.

54. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$.

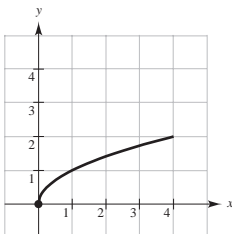
There is discontinuity at $x = 0$. To see that

$$\int_{-1}^1 \frac{|x|}{x} dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x -axis for $-1 \leq x \leq 1$. You see that the integral equals 0.

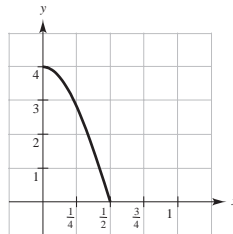


55.



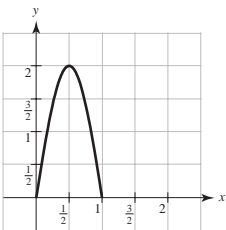
(a) $A \approx 5$ square units

56.



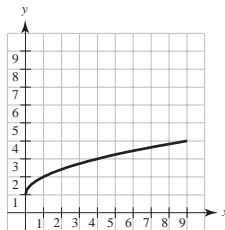
(b) $A \approx \frac{4}{3}$ square units

57.



(d) $\int_0^1 2 \sin \pi x dx \approx \frac{1}{2}(1)(2) \approx 1$

58.



(c) Area ≈ 27 .

59. $\int_0^3 x\sqrt{3-x} dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

60. $\int_0^3 \frac{5}{x^2+1} dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	6.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

61. $\int_0^{\pi/2} \sin^2 x \, dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

62. $\int_0^3 x \sin x \, dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

63. True

64. False

65. True

$$\int_0^1 x\sqrt{x} \, dx \neq \left(\int_0^1 x \, dx\right)\left(\int_0^1 \sqrt{x} \, dx\right)$$

66. True

67. False

68. True. The limits of integration are the same.

$$\int_0^2 (-x) \, dx = -2$$

69. $f(x) = x^2 + 3x, [0, 8]$

$$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$$

$$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$$

$$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x_i &= f(1) \Delta x_1 + f(2) \Delta x_2 + f(5) \Delta x_3 + f(8) \Delta x_4 \\ &= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272 \end{aligned}$$

70. $f(x) = \sin x, [0, 2\pi]$

$$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$$

$$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$$

$$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x_i &= f\left(\frac{\pi}{6}\right) \Delta x_1 + f\left(\frac{\pi}{3}\right) \Delta x_2 + f\left(\frac{2\pi}{3}\right) \Delta x_3 + f\left(\frac{3\pi}{2}\right) \Delta x_4 \\ &= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708 \end{aligned}$$

$$71. \Delta x = \frac{b-a}{n}, c_i = a + i(\Delta x) = a + i\left(\frac{b-a}{n}\right)$$

$$\begin{aligned} \int_a^b x \, dx &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[a + i\left(\frac{b-a}{n}\right) \right] \left(\frac{b-a}{n}\right) \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{b-a}{n}\right) \sum_{i=1}^n a + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^n i \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{b-a}{n} (an) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[a(b-a) + \frac{(b-a)^2 n+1}{2} \right] \\ &= a(b-a) + \frac{(b-a)^2}{2} \\ &= (b-a) \left[a + \frac{b-a}{2} \right] \\ &= \frac{(b-a)(a+b)}{2} = \frac{b^2 - a^2}{2} \end{aligned}$$

$$72. \Delta x = \frac{b-a}{n}, c_i = a + i(\Delta x) = a + i\left(\frac{b-a}{n}\right)$$

$$\begin{aligned} \int_a^b x^2 \, dx &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[a + i\left(\frac{b-a}{n}\right) \right]^2 \left(\frac{b-a}{n}\right) \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{b-a}{n}\right) \sum_{i=1}^n \left(a^2 + \frac{2ai(b-a)}{n} + i^2 \left(\frac{b-a}{n}\right)^2 \right) \right] \\ &= \lim_{n \rightarrow \infty} \left(\frac{b-a}{n}\right) \left[na^2 + \frac{2a(b-a)}{n} \frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)(2n+1)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[a^2(b-a) + \frac{a(b-a)^2(n+1)}{n} + \frac{(b-a)^3}{6} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3 \\ &= \frac{1}{3}(b^3 - a^3) \end{aligned}$$

$$73. f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

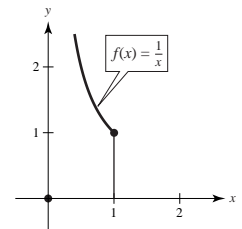
is not integrable on the interval $[0, 1]$. As $\|\Delta\| \rightarrow 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

$$74. f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & 0 < x \leq 1 \end{cases}$$

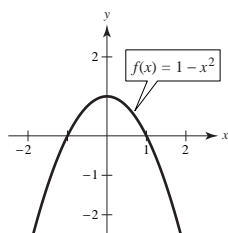
The limit

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

does not exist. This does not contradict Theorem 4.4 because f is not continuous on $[0, 1]$.



75. The function f is nonnegative between $x = -1$ and $x = 1$.

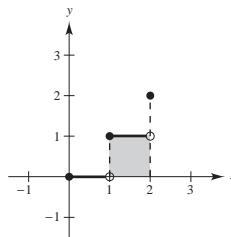


Hence,

$$\int_a^b (1 - x^2) dx$$

is a maximum for $a = -1$ and $b = 1$.

76. To find $\int_0^2 \llbracket x \rrbracket dx$, use a geometric approach.



Thus,

$$\int_0^2 \llbracket x \rrbracket dx = 1(2 - 1) = 1.$$

77. Let $f(x) = x^2$, $0 \leq x \leq 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

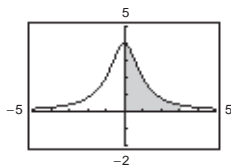
$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^3} [1^2 + 2^2 + 3^2 + \cdots + n^2] &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3} \end{aligned}$$

Section 4.4 The Fundamental Theorem of Calculus

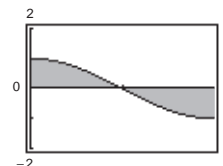
1. $f(x) = \frac{4}{x^2 + 1}$

$$\int_0^\pi \frac{4}{x^2 + 1} dx \text{ is positive.}$$



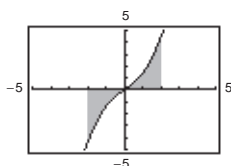
2. $f(x) = \cos x$

$$\int_0^\pi \cos x dx = 0$$



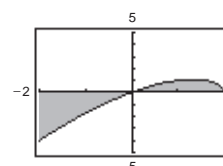
3. $f(x) = x\sqrt{x^2 + 1}$

$$\int_{-2}^2 x\sqrt{x^2 + 1} dx = 0$$



4. $f(x) = x\sqrt{2 - x}$

$$\int_{-2}^2 x\sqrt{2 - x} dx \text{ is negative.}$$



5. $\int_0^1 2x dx = \left[x^2\right]_0^1 = 1 - 0 = 1$

6. $\int_2^7 3 dv = \left[3v\right]_2^7 = 3(7) - 3(2) = 15$

7. $\int_{-1}^0 (x - 2) dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^0 = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$

8. $\int_2^5 (-3v + 4) v = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6 + 8) = -\frac{39}{2}$

9. $\int_{-1}^1 (t^2 - 2) dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^1 = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

$$10. \int_1^3 (3x^2 + 5x - 4) dx = \left[x^3 + \frac{5x^2}{2} - 4x \right]_1^3 = \left(27 + \frac{45}{2} - 12 \right) - \left(1 + \frac{5}{2} - 4 \right) = 38$$

$$11. \int_0^1 (2t - 1)^2 dt = \int_0^1 (4t^2 - 4t + 1) dt = \left[\frac{4}{3}t^3 - 2t^2 + t \right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$$

$$12. \int_{-1}^1 (t^3 - 9t) dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2 \right]_{-1}^1 = \left(\frac{1}{4} - \frac{9}{2} \right) - \left(\frac{1}{4} - \frac{9}{2} \right) = 0$$

$$13. \int_1^2 \left(\frac{3}{x^2} - 1 \right) dx = \left[-\frac{3}{x} - x \right]_1^2 = \left(-\frac{3}{2} - 2 \right) - (-3 - 1) = \frac{1}{2}$$

$$14. \int_{-2}^{-1} \left(u - \frac{1}{u^2} \right) du = \left[\frac{u^2}{2} + \frac{1}{u} \right]_{-2}^{-1} = \left(\frac{1}{2} - 1 \right) - \left(2 - \frac{1}{2} \right) = -2$$

$$15. \int_1^4 \frac{u-2}{\sqrt{u}} du = \int_1^4 (u^{1/2} - 2u^{-1/2}) du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2} \right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4} \right] - \left[\frac{2}{3} - 4 \right] = \frac{2}{3}$$

$$16. \int_{-3}^3 v^{1/3} dv = \left[\frac{3}{4}v^{4/3} \right]_{-3}^3 = \frac{3}{4}[(\sqrt[3]{-3})^4] - (\sqrt[3]{-3})^4 = 0$$

$$17. \int_{-1}^1 (\sqrt[3]{t} - 2) dt = \left[\frac{3}{4}t^{4/3} - 2t \right]_{-1}^1 = \left(\frac{3}{4} - 2 \right) - \left(\frac{3}{4} + 2 \right) = -4$$

$$18. \int_1^8 \sqrt{\frac{2}{x}} dx = \sqrt{2} \int_1^8 x^{-1/2} dx = \left[\sqrt{2}(2)x^{1/2} \right]_1^8 = \left[2\sqrt{2x} \right]_1^8 = 8 - 2\sqrt{2}$$

$$19. \int_0^1 \frac{x - \sqrt{x}}{3} dx = \frac{1}{3} \int_0^1 (x - x^{1/2}) dx = \frac{1}{3} \left[\frac{x^2}{2} - \frac{2}{3}x^{3/2} \right]_0^1 = \frac{1}{3} \left(\frac{1}{2} - \frac{2}{3} \right) = -\frac{1}{18}$$

$$20. \int_0^2 (2-t)\sqrt{t} dt = \int_0^2 (2t^{1/2} - t^{3/2}) dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2} \right]_0^2 = \left[\frac{t\sqrt{t}}{15}(20 - 6t) \right]_0^2 = \frac{2\sqrt{2}}{15}(20 - 12) = \frac{16\sqrt{2}}{15}$$

$$21. \int_{-1}^0 (t^{1/3} - t^{2/3}) dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3} \right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5} \right) = -\frac{27}{20}$$

$$22. \int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}} dx = \frac{1}{2} \int_{-8}^{-1} (x^{2/3} - x^{5/3}) dx = \frac{1}{2} \left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3} \right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24 - 15x) \right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$$

$$23. \int_0^3 |2x - 3| dx = \int_0^{3/2} (3 - 2x) dx + \int_{3/2}^3 (2x - 3) dx \quad \left(\text{split up the integral at the zero } x = \frac{3}{2} \right) = \left[3x - x^2 \right]_0^{3/2} + \left[x^2 - 3x \right]_{3/2}^3 = \left(\frac{9}{2} - \frac{9}{4} \right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2} \right) = 2 \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{9}{2}$$

$$\begin{aligned}
 24. \int_1^4 (3 - 1x - 31) dx &= \int_1^3 [3 + (x - 3)] dx + \int_3^4 [3 - (x - 3)] dx \\
 &= \int_1^3 x dx + \int_3^4 (6 - x) dx \\
 &= \left[\frac{x^2}{2} \right]_1^3 + \left[6x - \frac{x^2}{2} \right]_3^4 \\
 &= \left(\frac{9}{2} - \frac{1}{2} \right) + \left[(24 - 8) - \left(18 - \frac{9}{2} \right) \right] \\
 &= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}
 \end{aligned}$$

$$\begin{aligned}
 25. \int_0^3 |x^2 - 4| dx &= \int_0^2 (4 - x^2) dx + \int_2^3 (x^2 - 4) dx \\
 &= \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - 4x \right]_2^3 \\
 &= \left(8 - \frac{8}{3} \right) + (9 - 12) - \left(\frac{8}{3} - 8 \right) \\
 &= \frac{23}{3}
 \end{aligned}$$

$$\begin{aligned}
 26. \int_0^4 |x^2 - 4x + 3| dx &= \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^4 (x^2 - 4x + 3) dx \quad (\text{split up the integral at the} \\
 & \quad \text{zeros } x = 1, 3) \\
 &= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x \right]_3^4 \\
 &= \left(\frac{1}{3} - 2 + 3 \right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3 \right) + \left(\frac{64}{3} - 32 + 12 \right) - (9 - 18 + 9) \\
 &= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4
 \end{aligned}$$

$$27. \int_0^\pi (1 + \sin x) dx = \left[x - \cos x \right]_0^\pi = (\pi + 1) - (0 - 1) = 2 + \pi$$

$$28. \int_0^{\pi/4} \frac{1 - \sin^2 \theta}{\cos^2 \theta} d\theta = \int_0^{\pi/4} d\theta = \left[\theta \right]_0^{\pi/4} = \frac{\pi}{4} \qquad 29. \int_{-\pi/6}^{\pi/6} \sec^2 x dx = \left[\tan x \right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3} \right) = \frac{2\sqrt{3}}{3}$$

$$30. \int_{\pi/4}^{\pi/2} (2 - \csc^2 x) dx = \left[2x + \cot x \right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1 \right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$31. \int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta d\theta = \left[4 \sec \theta \right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$$

$$32. \int_{-\pi/2}^{\pi/2} (2t + \cos t) dt = \left[t^2 + \sin t \right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1 \right) - \left(\frac{\pi^2}{4} - 1 \right) = 2$$

$$33. A = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \qquad 34. A = \int_{-1}^1 (1 - x^4) dx = \left[x - \frac{1}{5}x^5 \right]_{-1}^1 = \frac{8}{5}$$

$$35. A = \int_0^3 (3 - x)\sqrt{x} dx = \int_0^3 (3x^{1/2} - x^{3/2}) dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x) \right]_0^3 = \frac{12\sqrt{3}}{5}$$

$$36. A = \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$37. A = \int_0^{\pi/2} \cos x dx = \left[\sin x \right]_0^{\pi/2} = 1$$

$$38. A = \int_0^{\pi} (x + \sin x) dx = \left[\frac{x^2}{2} - \cos x \right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$$

39. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1) dx = \left[x^3 + x \right]_0^2 = 8 + 2 = 10.$$

40. Since $y \geq 0$ on $[0, 8]$,

$$\text{Area} = \int_0^8 (1 + x^{1/3}) dx = \left[x + \frac{3}{4}x^{4/3} \right]_0^8 = 8 + \frac{3}{4}(16) = 20.$$

41. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x) dx = \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^2 = 4 + 2 = 6.$$

42. Since $y \geq 0$ on $[0, 3]$,

$$A = \int_0^3 (3x - x^2) dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3} \right]_0^3 = \frac{9}{2}.$$

$$43. \int_0^2 (x - 2\sqrt{x}) dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} \right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$(\sqrt{c} - 1)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}} \right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

$$44. \int_1^3 \frac{9}{x^3} dx = \left[-\frac{9}{2x^2} \right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

$$45. \int_{-\pi/4}^{\pi/4} 2 \sec^2 x dx = \left[2 \tan x \right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$$

$$f(c) \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 4$$

$$2 \sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm \frac{2}{\sqrt{\pi}}$$

$$c = \pm \operatorname{arcsec} \left(\frac{2}{\sqrt{\pi}} \right)$$

$$= \pm \arccos \frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

$$46. \int_{-\pi/3}^{\pi/3} \cos x dx = \left[\sin x \right]_{-\pi/3}^{\pi/3} = \sqrt{3}$$

$$f(c) \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \sqrt{3}$$

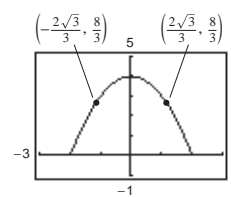
$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

$$47. \frac{1}{2 - (-2)} \int_{-2}^2 (4 - x^2) dx = \frac{1}{4} \left[4x - \frac{1}{3}x^3 \right]_{-2}^2 = \frac{1}{4} \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{8}{3}$$

$$\text{Average value} = \frac{8}{3}$$

$$4 - x^2 = \frac{8}{3} \text{ when } x^2 = 4 - \frac{8}{3} \text{ or } x = \pm \frac{2\sqrt{3}}{3} \approx \pm 1.155.$$



$$48. \frac{1}{3 - 1} \int_1^3 \frac{4(x^2 + 1)}{x^2} dx = 2 \int_1^3 (1 + x^{-2}) dx = 2 \left[x - \frac{1}{x} \right]_1^3 = 2 \left(3 - \frac{1}{3} \right) = \frac{16}{3}$$

$$\text{Average value} = \frac{16}{3}$$

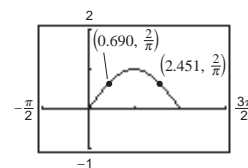
$$\frac{4(x^2 + 1)}{x^2} = \frac{16}{3} \Rightarrow x = \pm \sqrt{3}$$

$$49. \frac{1}{\pi - 0} \int_0^\pi \sin x dx = \left[-\frac{1}{\pi} \cos x \right]_0^\pi = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$

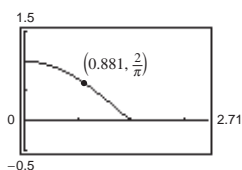


$$50. \frac{1}{(\pi/2) - 0} \int_0^{\pi/2} \cos x dx = \left[\frac{2}{\pi} \sin x \right]_0^{\pi/2} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$



51. The distance traveled is $\int_0^8 v(t) dt$. The area under the curve from $0 \leq t \leq 8$ is approximately (18 squares) $(30) \approx 540$ ft.

52. The distance traveled is $\int_0^5 v(t) dt$. The area under the curve from $0 \leq t \leq 5$ is approximately (29 squares) $(5) = 145$ ft.

53. If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$,

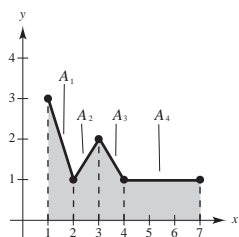
$$\text{then } \int_a^b f(x) dx = F(b) - F(a).$$

54. (a) $\int_1^7 f(x) dx = \text{Sum of the areas}$

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$$

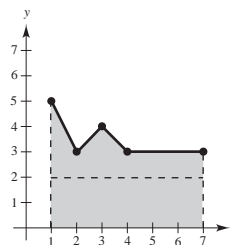
$$= 8$$



(b) Average value = $\frac{\int_1^7 f(x) dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



55. $\int_0^2 f(x) dx = -(\text{area of region A}) = -1.5$

56. $\int_2^6 f(x) dx = (\text{area of region B}) = \int_0^6 f(x) dx - \int_0^2 f(x) dx = 3.5 - (-1.5) = 5.0$

57. $\int_0^6 |f(x)| dx = -\int_0^2 f(x) dx + \int_2^6 f(x) dx = 1.5 + 5.0 = 6.5$

58. $\int_0^2 -2f(x) dx = -2 \int_0^2 f(x) dx = -2(-1.5) = 3.0$

59. $\int_0^6 [2 + f(x)] dx = \int_0^6 2 dx + \int_0^6 f(x) dx$
 $= 12 + 3.5 = 15.5$

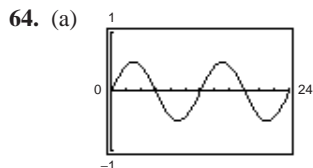
60. Average value $= \frac{1}{6} \int_0^6 f(x) dx = \frac{1}{6}(3.5) = 0.5833$

61. (a) $F(x) = k \sec^2 x$
 $F(0) = k = 500$
 $F(x) = 500 \sec^2 x$

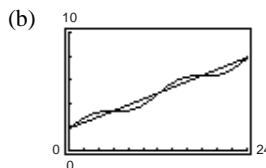
(b) $\frac{1}{\pi/3 - 0} \int_0^{\pi/3} 500 \sec^2 x dx = \frac{1500}{\pi} [\tan x]_0^{\pi/3}$
 $= \frac{1500}{\pi} (\sqrt{3} - 0)$
 ≈ 826.99 newtons
 ≈ 827 newtons

62. $\frac{1}{R-0} \int_0^R k(R^2 - r^2) dr = \frac{k}{R} [R^2 r - \frac{r^3}{3}]_0^R = \frac{2kR^2}{3}$

63. $\frac{1}{5-0} \int_0^5 (0.1729t + 0.1522t^2 - 0.0374t^3) dt \approx \frac{1}{5} [0.08645t^2 + 0.05073t^3 - 0.00935t^4]_0^5 \approx 0.5318$ liter

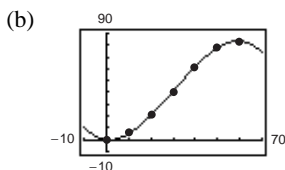


The area above the x -axis equals the area below the x -axis. Thus, the average value is zero.



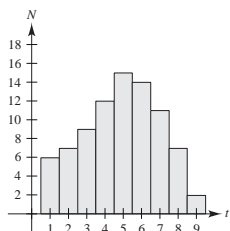
The average value of S appears to be g .

65. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$



(c) $\int_0^{60} v(t) dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t \right]_0^{60} \approx 2476$ meters

66. (a) histogram



(b) $[6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980$ customers

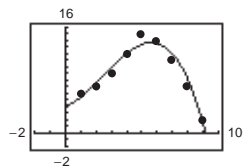
—CONTINUED—

66. —CONTINUED—

(c) Using a graphing utility, you obtain

$$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052t + 4.10317.$$

(d)



$$(e) \int_0^9 N(t) dt \approx 85.162$$

The estimated number of customers is $(85.162)(60) \approx 5110$.(f) Between 3 P.M. and 7 P.M., the number of customers is approximately $\left(\int_3^7 N(t) dt\right)(60) \approx (50.28)(60) \approx 3017$.Hence, $3017/240 \approx 12.6$ per minute.

$$67. F(x) = \int_0^x (t - 5) dt = \left[\frac{t^2}{2} - 5t\right]_0^x = \frac{x^2}{2} - 5x$$

$$F(2) = \frac{4}{2} - 5(2) = -8$$

$$F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$$

$$F(8) = \frac{64}{2} - 5(8) = -8$$

$$68. F(x) = \int_2^x (t^3 + 2t - 2) dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x$$

$$= \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4)$$

$$= \frac{x^4}{4} + x^2 - 2x - 4$$

$$F(2) = 4 + 4 - 4 - 4 = 0 \quad \left[\text{Note: } F(2) = \int_2^2 (t^3 + 2t - 2) dt = 0\right]$$

$$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$$

$$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$$

$$69. F(x) = \int_1^x \frac{10}{v^2} dv = \int_1^x 10v^{-2} dv = \left[\frac{-10}{v}\right]_1^x$$

$$= -\frac{10}{x} + 10 = 10\left(1 - \frac{1}{x}\right)$$

$$F(2) = 10\left(\frac{1}{2}\right) = 5$$

$$F(5) = 10\left(\frac{4}{5}\right) = 8$$

$$F(8) = 10\left(\frac{7}{8}\right) = \frac{35}{4}$$

$$70. F(x) = \int_2^x \frac{-2}{t^3} dt = -\int_2^x 2t^{-3} dt = \left[\frac{1}{t^2}\right]_2^x = \frac{1}{x^2} - \frac{1}{4}$$

$$F(2) = \frac{1}{4} - \frac{1}{4} = 0$$

$$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$$

$$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$$

$$71. F(x) = \int_1^x \cos \theta \, d\theta = \sin \theta \Big|_1^x = \sin x - \sin 1$$

$$F(2) = \sin 2 - \sin 1 = 0.0678$$

$$F(5) = \sin 5 - \sin 1 \approx -1.8004$$

$$F(8) = \sin 8 - \sin 1 \approx 0.1479$$

$$72. F(x) = \int_0^x \sin \theta \, d\theta = -\cos \theta \Big|_0^x = -\cos x + \cos 0 = 1 - \cos x$$

$$F(2) = 1 - \cos 2 \approx 1.4161$$

$$F(5) = 1 - \cos 5 \approx 0.7163$$

$$F(8) = 1 - \cos 8 \approx 1.1455$$

$$73. g(x) = \int_0^x f(t) \, dt$$

$$(a) g(0) = \int_0^0 f(t) \, dt = 0$$

$$g(2) = \int_0^2 f(t) \, dt \approx 4 + 2 + 1 = 7$$

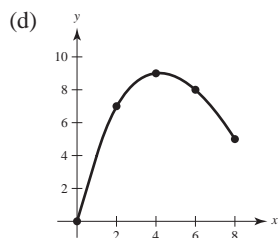
$$g(4) = \int_0^4 f(t) \, dt \approx 7 + 2 = 9$$

$$g(6) = \int_0^6 f(t) \, dt \approx 9 + (-1) = 8$$

$$g(8) = \int_0^8 f(t) \, dt \approx 8 - 3 = 5$$

(b) g increasing on $(0, 4)$ and decreasing on $(4, 8)$

(c) g is a maximum of 9 at $x = 4$.



$$74. g(x) = \int_0^x f(t) \, dt$$

$$(a) g(0) = \int_0^0 f(t) \, dt = 0$$

$$g(2) = \int_0^2 f(t) \, dt = -\frac{1}{2}(2)(4) = -4$$

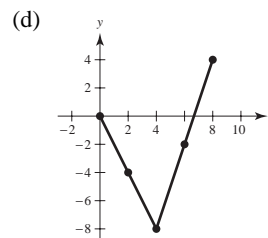
$$g(4) = \int_0^4 f(t) \, dt = -\frac{1}{2}(4)(4) = -8$$

$$g(6) = \int_0^6 f(t) \, dt = -8 + 2 + 4 = -2$$

$$g(8) = \int_0^8 f(t) \, dt = -2 + 6 = 4$$

(b) g decreasing on $(0, 4)$ and increasing on $(4, 8)$

(c) g is a minimum of -8 at $x = 4$.



$$75. (a) \int_0^x (t + 2) \, dt = \left[\frac{t^2}{2} + 2t \right]_0^x = \frac{1}{2}x^2 + 2x$$

$$(b) \frac{d}{dx} \left[\frac{1}{2}x^2 + 2x \right] = x + 2$$

$$76. (a) \int_0^x t(t^2 + 1) \, dt = \int_0^x (t^3 + t) \, dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2 \right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$$

$$(b) \frac{d}{dx} \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right] = x^3 + x = x(x^2 + 1)$$

$$77. (a) \int_8^x \sqrt[3]{t} \, dt = \left[\frac{3}{4}t^{4/3} \right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$$

$$(b) \frac{d}{dx} \left[\frac{3}{4}x^{4/3} - 12 \right] = x^{1/3} = \sqrt[3]{x}$$

$$78. (a) \int_4^x \sqrt{t} \, dt = \left[\frac{2}{3}t^{3/2} \right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$$

$$(b) \frac{d}{dx} \left[\frac{2}{3}x^{3/2} - \frac{16}{3} \right] = x^{1/2} = \sqrt{x}$$

$$79. (a) \int_{x/4}^x \sec^2 t \, dt = \left[\tan t \right]_{x/4}^x = \tan x - 1$$

$$(b) \frac{d}{dx}[\tan x - 1] = \sec^2 x$$

$$80. (a) \int_{\pi/3}^x \sec t \tan t \, dt = \left[\sec t \right]_{\pi/3}^x = \sec x - 2$$

$$(b) \frac{d}{dx}[\sec x - 2] = \sec x \tan x$$

$$81. F(x) = \int_{-2}^x (t^2 - 2t) \, dt$$

$$F'(x) = x^2 - 2x$$

$$82. F(x) = \int_1^x \frac{t^2}{t^2 + 1} \, dt$$

$$F'(x) = \frac{x^2}{x^2 + 1}$$

$$83. F(x) = \int_{-1}^x \sqrt{t^4 + 1} \, dt$$

$$F'(x) = \sqrt{x^4 + 1}$$

$$84. F(x) = \int_1^x \sqrt[4]{t} \, dt$$

$$F'(x) = \sqrt[4]{x}$$

$$85. F(x) = \int_0^x t \cos t \, dt$$

$$F'(x) = x \cos x$$

$$86. F(x) = \int_0^x \sec^3 t \, dt$$

$$F'(x) = \sec^3 x$$

$$87. F(x) = \int_x^{x+2} (4t + 1) \, dt$$

$$= \left[2t^2 + t \right]_x^{x+2}$$

$$= [2(x+2)^2 + (x+2)] - [2x^2 + x]$$

$$= 8x + 10$$

$$F'(x) = 8$$

Alternate solution:

$$F(x) = \int_x^{x+2} (4t + 1) \, dt$$

$$= \int_x^0 (4t + 1) \, dt + \int_0^{x+2} (4t + 1) \, dt$$

$$= -\int_0^x (4t + 1) \, dt + \int_0^{x+2} (4t + 1) \, dt$$

$$F'(x) = -(4x + 1) + 4(x + 2) + 1 = 8$$

$$88. F(x) = \int_{-x}^x t^3 \, dt = \left[\frac{t^4}{4} \right]_{-x}^x = 0$$

$$F'(x) = 0$$

Alternate solution:

$$F(x) = \int_{-x}^x t^3 \, dt$$

$$= \int_{-x}^0 t^3 \, dt + \int_0^x t^3 \, dt$$

$$= -\int_0^{-x} t^3 \, dt + \int_0^x t^3 \, dt$$

$$F'(x) = -(-x)^3(-1) + (x^3) = 0$$

$$89. F(x) = \int_0^{\sin x} \sqrt{t} \, dt = \left[\frac{2}{3} t^{3/2} \right]_0^{\sin x} = \frac{2}{3} (\sin x)^{3/2}$$

$$F'(x) = (\sin x)^{1/2} \cos x = \cos x \sqrt{\sin x}$$

Alternate solution:

$$F(x) = \int_0^{\sin x} \sqrt{t} \, dt$$

$$F'(x) = \sqrt{\sin x} \frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$$

$$90. F(x) = \int_2^{x^2} t^{-3} \, dt = \left[\frac{t^{-2}}{-2} \right]_2^{x^2} = \left[-\frac{1}{2t^2} \right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \Rightarrow F'(x) = 2x^{-5}$$

$$\text{Alternate solution: } F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$$

$$91. F(x) = \int_0^{x^3} \sin t^2 \, dt$$

$$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6$$

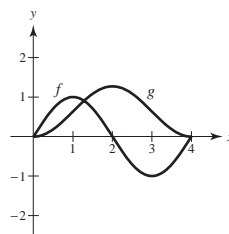
$$92. F(x) = \int_0^{x^2} \sin \theta^2 \, d\theta$$

$$F'(x) = \sin(x^2)^2 (2x) = 2x \sin x^4$$

$$93. g(x) = \int_0^x f(t) dt$$

$$g(0) = 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0$$

g has a relative maximum at $x = 2$.



94. (a)

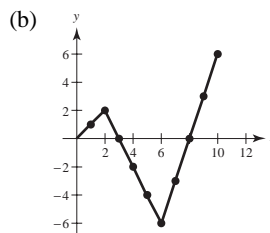
x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$, g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g : $x = 3, x = 8$



$$95. (a) C(x) = 5000 \left(25 + 3 \int_0^x t^{1/4} dt \right)$$

$$= 5000 \left(25 + 3 \left[\frac{4}{5} t^{5/4} \right]_0^x \right)$$

$$= 5000 \left(25 + \frac{12}{5} x^{5/4} \right) = 1000(125 + 12x^{5/4})$$

$$(b) C(1) = 1000(125 + 12(1)) = \$137,000$$

$$C(5) = 1000(125 + 12(5)^{5/4}) \approx \$214,721$$

$$C(10) = 1000(125 + 12(10)^{5/4}) \approx \$338,394$$

$$96. (a) g(t) = 4 - \frac{4}{t^2}$$

$$\lim_{t \rightarrow \infty} g(t) = 4$$

Horizontal asymptote: $y = 4$

$$(b) A(x) = \int_1^x \left(4 - \frac{4}{t^2} \right) dt$$

$$= \left[4t + \frac{4}{t} \right]_1^x = 4x + \frac{4}{x} - 8$$

$$= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$$

$$\lim_{x \rightarrow \infty} A(x) = \lim_{x \rightarrow \infty} \left(4x + \frac{4}{x} - 8 \right) = \infty + 0 - 8 = \infty$$

The graph of $A(x)$ does not have a horizontal asymptote.

$$97. x(t) = t^3 - 6t^2 + 9t - 2$$

$$x'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t-3)(t-1)$$

$$\text{Total distance} = \int_0^5 |x'(t)| dt$$

$$= \int_0^5 3|(t-3)(t-1)| dt$$

$$= 3 \int_0^1 (t^2 - 4t + 3) dt - 3 \int_1^3 (t^2 - 4t + 3) dt + 3 \int_3^5 (t^2 - 4t + 3) dt$$

$$= 4 + 4 + 20$$

$$= 28 \text{ units}$$

$$98. x(t) = (t - 1)(t - 3)^2 = t^3 - 7t^2 + 15t - 9$$

$$x'(t) = 3t^2 - 14t + 15$$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)| dt \approx 27.37 \text{ units.}$$

$$\begin{aligned} 99. \text{ Total distance} &= \int_1^4 |x'(t)| dt \\ &= \int_1^4 |v(t)| dt \\ &= \int_1^4 \frac{1}{\sqrt{t}} dt \\ &= 2t^{1/2} \Big|_1^4 \\ &= 2(2 - 1) = 2 \text{ units} \end{aligned}$$

$$100. P = \frac{2}{\pi} \int_0^{\pi/2} \sin \theta d\theta = \left[-\frac{2}{\pi} \cos \theta \right]_0^{\pi/2} = -\frac{2}{\pi}(0 - 1) = \frac{2}{\pi} \approx 63.7\%$$

101. True

102. True

103. The function $f(x) = x^{-2}$ is not continuous on $[-1, 1]$.

$$\int_{-1}^1 x^{-2} dx = \int_{-1}^0 x^{-2} dx + \int_0^1 x^{-2} dx$$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

104. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\begin{aligned} \int_{u(x)}^{v(x)} f(t) dt &= \left[F(t) \right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x)) \\ \frac{d}{dx} \left[\int_{u(x)}^{v(x)} f(t) dt \right] &= \frac{d}{dx} [F(v(x)) - F(u(x))] \\ &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

$$105. f(x) = \int_0^{1/x} \frac{1}{t^2 + 1} dt + \int_0^x \frac{1}{t^2 + 1} dt$$

By the Second Fundamental Theorem of Calculus, we have

$$\begin{aligned} f'(x) &= \frac{1}{(1/x)^2 + 1} \left(-\frac{1}{x^2} \right) + \frac{1}{x^2 + 1} \\ &= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0. \end{aligned}$$

Since $f'(x) = 0$, $f(x)$ must be constant.

$$106. G(x) = \int_0^x \left[s \int_0^s f(t) dt \right] ds$$

$$(a) G(0) = \int_0^0 \left[s \int_0^s f(t) dt \right] ds = 0$$

$$(c) G''(x) = x \cdot f(x) + \int_0^x f(t) dt$$

$$(d) G''(0) = 0 \cdot f(0) + \int_0^0 f(t) dt = 0$$

$$(b) \text{ Let } F(s) = \int_0^s f(t) dt.$$

$$G(x) = \int_0^x F(s) ds$$

$$G'(x) = F(x) = \int_0^x f(t) dt$$

$$G'(0) = 0 \int_0^0 f(t) dt = 0$$

Section 4.5 Integration by Substitution

$$\int f(g(x))g'(x) dx \quad u = g(x) \quad du = g'(x) dx$$

$$1. \int (5x^2 + 1)^2(10x) dx \quad 5x^2 + 1 \quad 10x dx$$

$$2. \int x^2 \sqrt{x^3 + 1} dx \quad x^3 + 1 \quad 3x^2 dx$$

$$3. \int \frac{x}{\sqrt{x^2 + 1}} dx \quad x^2 + 1 \quad 2x dx$$

$$4. \int \sec 2x \tan 2x dx \quad 2x \quad 2 dx$$

$$5. \int \tan^2 x \sec^2 x dx \quad \tan x \quad \sec^2 x dx$$

$$6. \int \frac{\cos x}{\sin^2 x} dx \quad \sin x \quad \cos x dx$$

$$7. \int (1 + 2x)^4(2) dx = \frac{(1 + 2x)^5}{5} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(1 + 2x)^5}{5} + C \right] = 2(1 + 2x)^4$$

$$8. \int (x^2 - 9)^3(2x) dx = \frac{(x^2 - 9)^4}{4} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^2 - 9)^4}{4} + C \right] = \frac{4(x^2 - 9)^3}{4}(2x) = (x^2 - 9)^3(2x)$$

$$9. \int (9 - x^2)^{1/2}(-2x) dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{2}{3}(9 - x^2)^{3/2} + C \right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$$

$$10. \int (1 - 2x^2)^{1/3}(-4x) dx = \frac{3}{4}(1 - 2x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{3}{4}(1 - 2x^2)^{4/3} + C \right] = \frac{3}{4} \cdot \frac{4}{3}(1 - 2x^2)^{1/3}(-4x) = (1 - 2x^2)^{1/3}(-4x)$$

$$11. \int x^3(x^4 + 3)^2 dx = \frac{1}{4} \int (x^4 + 3)^2(4x^3) dx = \frac{1}{4} \frac{(x^4 + 3)^3}{3} + C = \frac{(x^4 + 3)^3}{12} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^4 + 3)^3}{12} + C \right] = \frac{3(x^4 + 3)^2}{12}(4x^3) = (x^4 + 3)^2(x^3)$$

$$12. \int x^2(x^3 + 5)^4 dx = \frac{1}{3} \int (x^3 + 5)^4(3x^2) dx = \frac{1}{3} \frac{(x^3 + 5)^5}{5} + C = \frac{(x^3 + 5)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 + 5)^5}{15} + C \right] = \frac{5(x^3 + 5)^4(3x^2)}{15} = (x^3 + 5)^4 x^2$$

$$13. \int x^2(x^3 - 1)^4 dx = \frac{1}{3} \int (x^3 - 1)^4(3x^2) dx = \frac{1}{3} \left[\frac{(x^3 - 1)^5}{5} \right] + C = \frac{(x^3 - 1)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 - 1)^5}{15} + C \right] = \frac{5(x^3 - 1)^4(3x^2)}{15} = x^2(x^3 - 1)^4$$

$$14. \int x(4x^2 + 3)^3 dx = \frac{1}{8} \int (4x^2 + 3)^3(8x) dx = \frac{1}{8} \left[\frac{(4x^2 + 3)^4}{4} \right] + C = \frac{(4x^2 + 3)^4}{32} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(4x^2 + 3)^4}{32} + C \right] = \frac{4(4x^2 + 3)^3(8x)}{32} = x(4x^2 + 3)^3$$

$$15. \int t\sqrt{t^2 + 2} dt = \frac{1}{2} \int (t^2 + 2)^{1/2}(2t) dt = \frac{1}{2} \frac{(t^2 + 2)^{3/2}}{3/2} + C = \frac{(t^2 + 2)^{3/2}}{3} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{(t^2 + 2)^{3/2}}{3} + C \right] = \frac{3/2(t^2 + 2)^{1/2}(2t)}{3} = (t^2 + 2)^{1/2}t$$

$$16. \int t^3\sqrt{t^4 + 5} dt = \frac{1}{4} \int (t^4 + 5)^{1/2}(4t^3) dt = \frac{1}{4} \frac{(t^4 + 5)^{3/2}}{3/2} + C = \frac{1}{6}(t^4 + 5)^{3/2} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{6}(t^4 + 5)^{3/2} + C \right] = \frac{1}{6} \cdot \frac{3}{2}(t^4 + 5)^{1/2}(4t^3) = (t^4 + 5)^{1/2}(t^3)$$

$$17. \int 5x(1 - x^2)^{1/3} dx = -\frac{5}{2} \int (1 - x^2)^{1/3}(-2x) dx = -\frac{5}{2} \cdot \frac{(1 - x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1 - x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{15}{8}(1 - x^2)^{4/3} + C \right] = -\frac{15}{8} \cdot \frac{4}{3}(1 - x^2)^{1/3}(-2x) = 5x(1 - x^2)^{1/3} = 5x\sqrt[3]{1 - x^2}$$

$$18. \int u^2\sqrt{u^3 + 2} du = \frac{1}{3} \int (u^3 + 2)^{1/2}(3u^2) du = \frac{1}{3} \frac{(u^3 + 2)^{3/2}}{3/2} + C = \frac{2(u^3 + 2)^{3/2}}{9} + C$$

$$\text{Check: } \frac{d}{du} \left[\frac{2(u^3 + 2)^{3/2}}{9} + C \right] = \frac{2}{9} \cdot \frac{3}{2}(u^3 + 2)^{1/2}(3u^2) = (u^3 + 2)^{1/2}(u^2)$$

$$19. \int \frac{x}{(1 - x^2)^3} dx = -\frac{1}{2} \int (1 - x^2)^{-3}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{-2}}{-2} + C = \frac{1}{4(1 - x^2)^2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{4(1 - x^2)^2} + C \right] = \frac{1}{4}(-2)(1 - x^2)^{-3}(-2x) = \frac{x}{(1 - x^2)^3}$$

$$20. \int \frac{x^3}{(1 + x^4)^2} dx = \frac{1}{4} \int (1 + x^4)^{-2}(4x^3) dx = -\frac{1}{4}(1 + x^4)^{-1} + C = \frac{-1}{4(1 + x^4)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{-1}{4(1 + x^4)} + C \right] = \frac{1}{4}(1 + x^4)^{-2}(4x^3) = \frac{x^3}{(1 + x^4)^2}$$

$$21. \int \frac{x^2}{(1 + x^3)^2} dx = \frac{1}{3} \int (1 + x^3)^{-2}(3x^2) dx = \frac{1}{3} \left[\frac{(1 + x^3)^{-1}}{-1} \right] + C = -\frac{1}{3(1 + x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{3(1 + x^3)} + C \right] = -\frac{1}{3}(-1)(1 + x^3)^{-2}(3x^2) = \frac{x^2}{(1 + x^3)^2}$$

$$22. \int \frac{x^2}{(16 - x^3)^2} dx = -\frac{1}{3} \int (16 - x^3)^{-2}(-3x^2) dx = -\frac{1}{3} \left[\frac{(16 - x^3)^{-1}}{-1} \right] + C = \frac{1}{3(16 - x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3(16 - x^3)} + C \right] = \frac{1}{3}(-1)(16 - x^3)^{-2}(3x^2) = \frac{x^2}{(16 - x^3)^2}$$

$$23. \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int (1-x^2)^{-1/2} (-2x) dx = -\frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$$

$$\text{Check: } \frac{d}{dx} [-(1-x^2)^{1/2} + C] = -\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$$

$$24. \int \frac{x^3}{\sqrt{1+x^4}} dx = \frac{1}{4} \int (1+x^4)^{-1/2} (4x^3) dx = \frac{1}{4} \frac{(1+x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{\sqrt{1+x^4}}{2} + C \right] = \frac{1}{2} \cdot \frac{1}{2} (1+x^4)^{-1/2} (4x^3) = \frac{x^3}{\sqrt{1+x^4}}$$

$$25. \int \left(1 + \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt = -\int \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) dt = -\frac{[1 + (1/t)]^4}{4} + C$$

$$\text{Check: } \frac{d}{dt} \left[-\frac{[1 + (1/t)]^4}{4} + C \right] = -\frac{1}{4}(4) \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) = \frac{1}{t^2} \left(1 + \frac{1}{t}\right)^3$$

$$26. \int \left[x^2 + \frac{1}{(3x)^2} \right] dx = \int \left(x^2 + \frac{1}{9} x^{-2} \right) dx = \frac{x^3}{3} + \frac{1}{9} \left(\frac{x^{-1}}{-1} \right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4 - 1}{9x} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3} x^3 - \frac{1}{9} x^{-1} + C \right] = x^2 + \frac{1}{9} x^{-2} = x^2 + \frac{1}{(3x)^2}$$

$$27. \int \frac{1}{\sqrt{2x}} dx = \frac{1}{2} \int (2x)^{-1/2} 2 dx = \frac{1}{2} \left[\frac{(2x)^{1/2}}{1/2} \right] + C = \sqrt{2x} + C$$

$$\text{Alternate Solution: } \int \frac{1}{\sqrt{2x}} dx = \frac{1}{\sqrt{2}} \int x^{-1/2} dx = \frac{1}{\sqrt{2}} \frac{x^{1/2}}{(1/2)} + C = \sqrt{2x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{2x} + C] = \frac{1}{2} (2x)^{-1/2} (2) = \frac{1}{\sqrt{2x}}$$

$$28. \int \frac{1}{2\sqrt{x}} dx = \frac{1}{2} \int x^{-1/2} dx = \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C = \sqrt{x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{x} + C] = \frac{1}{2\sqrt{x}}$$

$$29. \int \frac{x^2 + 3x + 7}{\sqrt{x}} dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2}) dx = \frac{2}{5} x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5} \sqrt{x}(x^2 + 5x + 35) + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{2}{5} x^{5/2} + 2x^{3/2} + 14x^{1/2} + C \right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$$

$$30. \int \frac{t + 2t^2}{\sqrt{t}} dt = \int (t^{1/2} + 2t^{3/2}) dt = \frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C = \frac{2}{15} t^{3/2} (5 + 6t) + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C \right] = t^{1/2} + 2t^{3/2} = \frac{t + 2t^2}{\sqrt{t}}$$

$$31. \int t^2 \left(t - \frac{2}{t} \right) dt = \int (t^3 - 2t) dt = \frac{1}{4} t^4 - t^2 + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{4} t^4 - t^2 + C \right] = t^3 - 2t = t^2 \left(t - \frac{2}{t} \right)$$

$$32. \int \left(\frac{t^3}{3} + \frac{1}{4t^2} \right) dt = \int \left(\frac{1}{3}t^3 + \frac{1}{4}t^{-2} \right) dt = \frac{1}{3} \left(\frac{t^4}{4} \right) + \frac{1}{4} \left(\frac{t^{-1}}{-1} \right) + C = \frac{1}{12}t^4 - \frac{1}{4t} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{12}t^4 - \frac{1}{4t} + C \right] = \frac{1}{3}t^3 + \frac{1}{4t^2}$$

$$33. \int (9 - y)\sqrt{y} dy = \int (9y^{1/2} - y^{3/2}) dy = 9 \left(\frac{2}{3}y^{3/2} \right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$$

$$\text{Check: } \frac{d}{dy} \left[\frac{2}{5}y^{3/2}(15 - y) + C \right] = \frac{d}{dy} \left[6y^{3/2} - \frac{2}{5}y^{5/2} + C \right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$$

$$34. \int 2\pi y(8 - y^{3/2}) dy = 2\pi \int (8y - y^{5/2}) dy = 2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$$

$$\text{Check: } \frac{d}{dy} \left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C \right] = \frac{d}{dy} \left[2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C \right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$$

$$35. y = \int \left[4x + \frac{4x}{\sqrt{16 - x^2}} \right] dx$$

$$= 4 \int x dx - 2 \int (16 - x^2)^{-1/2} (-2x) dx$$

$$= 4 \left(\frac{x^2}{2} \right) - 2 \left[\frac{(16 - x^2)^{1/2}}{1/2} \right] + C$$

$$= 2x^2 - 4\sqrt{16 - x^2} + C$$

$$36. y = \int \frac{10x^2}{\sqrt{1 + x^3}} dx$$

$$= \frac{10}{3} \int (1 + x^3)^{-1/2} (3x^2) dx$$

$$= \frac{10}{3} \left[\frac{(1 + x^3)^{1/2}}{1/2} \right] + C$$

$$= \frac{20}{3} \sqrt{1 + x^3} + C$$

$$37. y = \int \frac{x + 1}{(x^2 + 2x - 3)^2} dx$$

$$= \frac{1}{2} \int (x^2 + 2x - 3)^{-2} (2x + 2) dx$$

$$= \frac{1}{2} \left[\frac{(x^2 + 2x - 3)^{-1}}{-1} \right] + C$$

$$= -\frac{1}{2(x^2 + 2x - 3)} + C$$

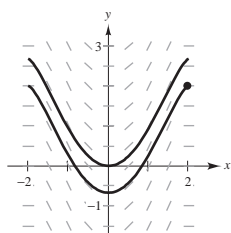
$$38. y = \int \frac{x - 4}{\sqrt{x^2 - 8x + 1}} dx$$

$$= \frac{1}{2} \int (x^2 - 8x + 1)^{-1/2} (2x - 8) dx$$

$$= \frac{1}{2} \left[\frac{(x^2 - 8x + 1)^{1/2}}{1/2} \right] + C$$

$$= \sqrt{x^2 - 8x + 1} + C$$

39. (a)



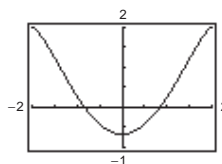
(b) $\frac{dy}{dx} = x\sqrt{4 - x^2}, (2, 2)$

$$y = \int x\sqrt{4 - x^2} dx = -\frac{1}{2} \int (4 - x^2)^{1/2} (-2x dx)$$

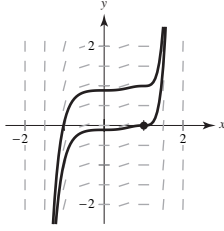
$$= -\frac{1}{2} \cdot \frac{2}{3} (4 - x^2)^{3/2} + C = -\frac{1}{3} (4 - x^2)^{3/2} + C$$

$$(2, 2): 2 = -\frac{1}{3}(4 - 2^2)^{3/2} + C \Rightarrow C = 2$$

$$y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$$



40. (a)



(b) $\frac{dy}{dx} = x^2(x^3 - 1)^2, (1, 0)$

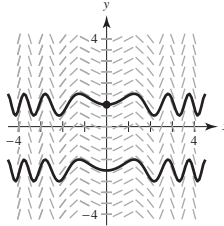
$$y = \int x^2(x^3 - 1)^2 dx = \frac{1}{3} \int (x^3 - 1)^2 (3x^2 dx) \quad (u = x^3 - 1)$$

$$= \frac{1}{3} \frac{(x^3 - 1)^3}{3} + C = \frac{1}{9}(x^3 - 1)^3 + C$$

$$0 = C$$

$$y = \frac{1}{9}(x^3 - 1)^3$$

41. (a)



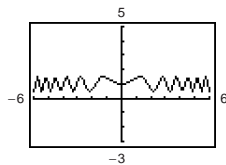
(b) $\frac{dy}{dx} = x \cos x^2, (0, 1)$

$$y = \int x \cos x^2 dx = \frac{1}{2} \int \cos(x^2) 2x dx$$

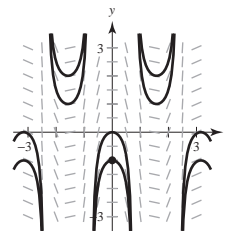
$$= \frac{1}{2} \sin(x^2) + C$$

$$(0, 1): 1 = \frac{1}{2} \sin(0) + C \Rightarrow C = 1$$

$$y = \frac{1}{2} \sin(x^2) + 1$$



42. (a)



(b) $\frac{dy}{dx} = -2 \sec(2x) \tan(2x), (0, -1)$

$$y = \int -2 \sec(2x) \tan(2x) dx \quad (u = 2x)$$

$$= -\sec(2x) + C$$

$$-1 = -\sec(0) + C \Rightarrow C = 0$$

$$y = -\sec(2x)$$

43. $\int \pi \sin \pi x dx = -\cos \pi x + C$

44. $\int 4x^3 \sin x^4 dx = \int \sin x^4 (4x^3) dx = -\cos x^4 + C$

45. $\int \sin 2x dx = \frac{1}{2} \int (\sin 2x)(2x) dx = -\frac{1}{2} \cos 2x + C$

46. $\int \cos 6x dx = \frac{1}{6} \int (\cos 6x)(6) dx = \frac{1}{6} \sin 6x + C$

47. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta = -\int \cos \frac{1}{\theta} \left(-\frac{1}{\theta^2}\right) d\theta = -\sin \frac{1}{\theta} + C$

48. $\int x \sin x^2 dx = \frac{1}{2} \int (\sin x^2)(2x) dx = -\frac{1}{2} \cos x^2 + C$

49. $\int \sin 2x \cos 2x dx = \frac{1}{2} \int (\sin 2x)(2 \cos 2x) dx = \frac{1}{2} \frac{(\sin 2x)^2}{2} + C = \frac{1}{4} \sin^2 2x + C$ OR

$$\int \sin 2x \cos 2x dx = -\frac{1}{2} \int (\cos 2x)(-2 \sin 2x) dx = -\frac{1}{2} \frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4} \cos^2 2x + C_1$$
 OR

$$\int \sin 2x \cos 2x dx = \frac{1}{2} \int 2 \sin 2x \cos 2x dx = \frac{1}{2} \int \sin 4x dx = -\frac{1}{8} \cos 4x + C_2$$

$$50. \int \sec(1-x) \tan(1-x) dx = - \int [\sec(1-x) \tan(1-x)](-1) dx = -\sec(1-x) + C$$

$$51. \int \tan^4 x \sec^2 x dx = \frac{\tan^5 x}{5} + C = \frac{1}{5} \tan^5 x + C$$

$$52. \int \sqrt{\tan x} \sec^2 x dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3} (\tan x)^{3/2} + C$$

$$53. \int \frac{\csc^2 x}{\cot^3 x} dx = - \int (\cot x)^{-3} (-\csc^2 x) dx \\ = -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2 \cot^2 x} + C = \frac{1}{2} \tan^2 x + C = \frac{1}{2} (\sec^2 x - 1) + C = \frac{1}{2} \sec^2 x + C_1$$

$$54. \int \frac{\sin x}{\cos^3 x} dx = - \int (\cos x)^{-3} (-\sin x) dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2 \cos^2 x} + C = \frac{1}{2} \sec^2 x + C$$

$$55. \int \cot^2 x dx = \int (\csc^2 x - 1) dx = -\cot x - x + C$$

$$56. \int \csc^2\left(\frac{x}{2}\right) dx = 2 \int \csc^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = -2 \cot\left(\frac{x}{2}\right) + C$$

$$57. f(x) = \int \cos \frac{x}{2} dx = 2 \sin \frac{x}{2} + C$$

Since $f(0) = 3 = 2 \sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2 \sin \frac{x}{2} + 3.$$

$$58. f(x) = \int \pi \sec \pi x \tan \pi x dx = \sec \pi x + C$$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$$f(x) = \sec \pi x - 1.$$

$$59. f'(x) = \sin 4x, \left(\frac{\pi}{4}, \frac{-3}{4}\right)$$

$$f(x) = \frac{-1}{4} \cos 4x + C$$

$$f\left(\frac{\pi}{4}\right) = \frac{-1}{4} \cos\left(4\left(\frac{\pi}{4}\right)\right) + C = \frac{-3}{4}$$

$$-\frac{1}{4}(-1) + C = \frac{-3}{4}$$

$$C = -1$$

$$f(x) = -\frac{1}{4} \cos 4x - 1$$

$$60. f'(x) = \sec^2(2x), \left(\frac{\pi}{2}, 2\right)$$

$$f(x) = \frac{1}{2} \tan(2x) + C$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2} \tan\left(2\left(\frac{\pi}{2}\right)\right) + C = 2$$

$$\frac{1}{2}(0) + C = 2$$

$$C = 2$$

$$f(x) = \frac{1}{2} \tan(2x) + 2$$

$$61. f'(x) = 2x(4x^2 - 10)^2, (2, 10)$$

$$f(x) = \frac{(4x^2 - 10)^3}{12} + C = \frac{2(2x^2 - 5)^3}{3} + C$$

$$f(2) = \frac{2(8 - 5)^3}{3} + C = 18 + C = 10 \Rightarrow C = -8$$

$$f(x) = \frac{2}{3}(2x^2 - 5)^3 - 8$$

$$62. f'(x) = -2x\sqrt{8 - x^2}, (2, 7)$$

$$f(x) = \frac{2(8 - x^2)^{3/2}}{3} + C$$

$$f(2) = \frac{2(4)^{3/2}}{3} + C = \frac{16}{3} + C = 7 \Rightarrow C = \frac{5}{3}$$

$$f(x) = \frac{2(8 - x^2)^{3/2}}{3} + \frac{5}{3}$$

63. $u = x + 2, x = u - 2, dx = du$

$$\begin{aligned} \int x\sqrt{x+2} dx &= \int (u-2)\sqrt{u} du \\ &= \int (u^{3/2} - 2u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\ &= \frac{2u^{3/2}}{15}(3u - 10) + C \\ &= \frac{2}{15}(x+2)^{3/2}[3(x+2) - 10] + C \\ &= \frac{2}{15}(x+2)^{3/2}(3x-4) + C \end{aligned}$$

64. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2} du$

$$\begin{aligned} \int x\sqrt{2x+1} dx &= \int \frac{1}{2}(u-1)\sqrt{u} \frac{1}{2} du \\ &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) du \\ &= \frac{1}{4} \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right) + C \\ &= \frac{u^{3/2}}{30}(3u - 5) + C \\ &= \frac{1}{30}(2x+1)^{3/2}[3(2x+1) - 5] + C \\ &= \frac{1}{30}(2x+1)^{3/2}(6x-2) + C \\ &= \frac{1}{15}(2x+1)^{3/2}(3x-1) + C \end{aligned}$$

65. $u = 1 - x, x = 1 - u, dx = -du$

$$\begin{aligned} \int x^2\sqrt{1-x} dx &= - \int (1-u)^2\sqrt{u} du \\ &= - \int (u^{1/2} - 2u^{3/2} + u^{5/2}) du \\ &= - \left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2} \right) + C \\ &= - \frac{2u^{3/2}}{105}(35 - 42u + 15u^2) + C \\ &= - \frac{2}{105}(1-x)^{3/2}[35 - 42(1-x) + 15(1-x)^2] + C \\ &= - \frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C \end{aligned}$$

66. $u = 2 - x, x = 2 - u, dx = -du$

$$\begin{aligned} \int (x+1)\sqrt{2-x} dx &= - \int (3-u)\sqrt{u} du \\ &= - \int (3u^{1/2} - u^{3/2}) du \\ &= - \left(2u^{3/2} - \frac{2}{5}u^{5/2} \right) + C \\ &= - \frac{2u^{3/2}}{5}(5 - u) + C \\ &= - \frac{2}{5}(2-x)^{3/2}[5 - (2-x)] + C \\ &= - \frac{2}{5}(2-x)^{3/2}(x+3) + C \end{aligned}$$

67. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2} du$

$$\begin{aligned} \int \frac{x^2 - 1}{\sqrt{2x - 1}} dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}} \frac{1}{2} du \\ &= \frac{1}{8} \int u^{-1/2} [(u^2 + 2u + 1) - 4] du \\ &= \frac{1}{8} \int (u^{3/2} + 2u^{1/2} - 3u^{-1/2}) du \\ &= \frac{1}{8} \left(\frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} - 6u^{1/2} \right) + C \\ &= \frac{u^{1/2}}{60} (3u^2 + 10u - 45) + C \\ &= \frac{\sqrt{2x - 1}}{60} [3(2x - 1)^2 + 10(2x - 1) - 45] + C \\ &= \frac{1}{60} \sqrt{2x - 1} (12x^2 + 8x - 52) + C \\ &= \frac{1}{15} \sqrt{2x - 1} (3x^2 + 2x - 13) + C \end{aligned}$$

68. Let $u = x + 4, x = u - 4, du = dx$.

$$\begin{aligned} \int \frac{2x + 1}{\sqrt{x + 4}} dx &= \int \frac{2(u - 4) + 1}{\sqrt{u}} du \\ &= \int (2u^{1/2} - 7u^{-1/2}) du \\ &= \frac{4}{3} u^{3/2} - 14u^{1/2} + C \\ &= \frac{2}{3} u^{1/2} (2u - 21) + C \\ &= \frac{2}{3} \sqrt{x + 4} [2(x + 4) - 21] + C \\ &= \frac{2}{3} \sqrt{x + 4} (2x - 13) + C \end{aligned}$$

69. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned} \int \frac{-x}{(x + 1) - \sqrt{x + 1}} dx &= \int \frac{-(u - 1)}{u - \sqrt{u}} du \\ &= - \int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)} du \\ &= - \int (1 + u^{-1/2}) du \\ &= -(u + 2u^{1/2}) + C \\ &= -u - 2\sqrt{u} + C \\ &= -(x + 1) - 2\sqrt{x + 1} + C \\ &= -x - 2\sqrt{x + 1} - 1 + C \\ &= -(x + 2\sqrt{x + 1}) + C_1 \end{aligned}$$

where $C_1 = -1 + C$.

70. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned} \int t \sqrt[3]{t - 4} dt &= \int (u + 4) u^{1/3} du \\ &= \int (u^{4/3} + 4u^{1/3}) du \\ &= \frac{3}{7} u^{7/3} + 3u^{4/3} + C \\ &= \frac{3u^{4/3}}{7} (u + 7) + C \\ &= \frac{3}{7} (t - 4)^{4/3} [(t - 4) + 7] + C \\ &= \frac{3}{7} (t - 4)^{4/3} (t + 3) + C \end{aligned}$$

71. Let $u = x^2 + 1$, $du = 2x dx$.

$$\int_{-1}^1 x(x^2 + 1)^3 dx = \frac{1}{2} \int_{-1}^1 (x^2 + 1)^3 (2x) dx = \left[\frac{1}{8}(x^2 + 1)^4 \right]_{-1}^1 = 0$$

72. Let $u = x^3 + 8$, $du = 3x^2 dx$.

$$\begin{aligned} \int_{-2}^4 x^2(x^3 + 8)^2 dx &= \frac{1}{3} \int_{-2}^4 (x^3 + 8)^2 (3x^2) dx = \left[\frac{1}{3} \frac{(x^3 + 8)^3}{3} \right]_{-2}^4 \\ &= \frac{1}{9} [(64 + 8)^3 - (-8 + 8)^3] = 41,472 \end{aligned}$$

73. Let $u = x^3 + 1$, $du = 3x^2 dx$.

$$\begin{aligned} \int_1^2 2x^2 \sqrt{x^3 + 1} dx &= 2 \cdot \frac{1}{3} \int_1^2 (x^3 + 1)^{1/2} (3x^2) dx \\ &= \left[\frac{2}{3} \frac{(x^3 + 1)^{3/2}}{3/2} \right]_1^2 \\ &= \frac{4}{9} [(x^3 + 1)^{3/2}]_1^2 \\ &= \frac{4}{9} [27 - 2\sqrt{2}] = 12 - \frac{8}{9}\sqrt{2} \end{aligned}$$

74. Let $u = 1 - x^2$, $du = -2x dx$.

$$\int_0^1 x \sqrt{1 - x^2} dx = -\frac{1}{2} \int_0^1 (1 - x^2)^{1/2} (-2x) dx = \left[-\frac{1}{3}(1 - x^2)^{3/2} \right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

75. Let $u = 2x + 1$, $du = 2 dx$.

$$\int_0^4 \frac{1}{\sqrt{2x + 1}} dx = \frac{1}{2} \int_0^4 (2x + 1)^{-1/2} (2) dx = \left[\sqrt{2x + 1} \right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

76. Let $u = 1 + 2x^2$, $du = 4x dx$.

$$\int_0^2 \frac{x}{\sqrt{1 + 2x^2}} dx = \frac{1}{4} \int_0^2 (1 + 2x^2)^{-1/2} (4x) dx = \left[\frac{1}{2} \sqrt{1 + 2x^2} \right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

77. Let $u = 1 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int_1^9 \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx = 2 \int_1^9 (1 + \sqrt{x})^{-2} \left(\frac{1}{2\sqrt{x}} \right) dx = \left[-\frac{2}{1 + \sqrt{x}} \right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

78. Let $u = 4 + x^2$, $du = 2x dx$.

$$\int_0^2 x \sqrt[3]{4 + x^2} dx = \frac{1}{2} \int_0^2 (4 + x^2)^{1/3} (2x) dx = \left[\frac{3}{8} (4 + x^2)^{4/3} \right]_0^2 = \frac{3}{8} (8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2} \sqrt[3]{4} \approx 3.619$$

79. $u = 2 - x$, $x = 2 - u$, $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x - 1) \sqrt{2 - x} dx = \int_1^0 -[(2 - u) - 1] \sqrt{u} du = \int_1^0 (u^{3/2} - u^{1/2}) du = \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3} \right] = \frac{4}{15}$$

80. Let $u = 2x - 1$, $du = 2 dx$, $x = \frac{1}{2}(u + 1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}} dx &= \int_1^9 \frac{1/2(u+1)}{\sqrt{u}} \frac{1}{2} du = \frac{1}{4} \int_1^9 (u^{1/2} + u^{-1/2}) du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{3/2} + 2u^{1/2} \right]_1^9 \\ &= \frac{1}{4} \left[\left(\frac{2}{3}(27) + 2(3) \right) - \left(\frac{2}{3} + 2 \right) \right] \\ &= \frac{16}{3} \end{aligned}$$

81. $\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right) dx = \left[\frac{3}{2} \sin\left(\frac{2}{3}x\right) \right]_0^{\pi/2} = \frac{3}{2} \left(\frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}}{4}$

82. $\int_{\pi/3}^{\pi/2} (x + \cos x) dx = \left[\frac{x^2}{2} + \sin x \right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1 \right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2} \right) = \frac{5\pi^2}{72} + \frac{2 - \sqrt{3}}{2}$

83. $\frac{dy}{dx} = 18x^2(2x^3 + 1)^2$, $(0, 4)$

$$y = 3 \int (2x^3 + 1)^2 (6x^2) dx \quad (u = 2x^3 + 1)$$

$$y = 3 \frac{(2x^3 + 1)^3}{3} + C = (2x^3 + 1)^3 + C$$

$$4 = 1^3 + C \Rightarrow C = 3$$

$$y = (2x^3 + 1)^3 + 3$$

84. $\frac{dy}{dx} = \frac{-48}{(3x+5)^3}$, $(-1, 3)$

$$y = -48 \int (3x+5)^{-3} dx$$

$$= (-48) \frac{1}{3} \int (3x+5)^{-3} 3 dx$$

$$= \frac{-16(3x+5)^{-2}}{-2} + C$$

$$= \frac{8}{(3x+5)^2} + C$$

$$3 = \frac{8}{(3(-1)+5)^2} + C = \frac{8}{4} + C \Rightarrow C = 1$$

$$y = \frac{8}{(3x+5)^2} + 1$$

85. $\frac{dy}{dx} = \frac{2x}{\sqrt{2x^2-1}}$, $(5, 4)$

$$y = \frac{1}{2} \int (2x^2-1)^{-1/2} (4x) dx \quad (u = 2x^2-1)$$

$$y = \frac{1}{2} \frac{(2x^2-1)^{1/2}}{1/2} + C = \sqrt{2x^2-1} + C$$

$$4 = \sqrt{49} + C = 7 + C \Rightarrow C = -3$$

$$y = \sqrt{2x^2-1} - 3$$

86. $\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3+1)^{3/2}}$, $(0, 2)$

$$y = \int (4x + (3x^3+1)^{-3/2} 9x^2) dx$$

$$= 2x^2 + \frac{(3x^3+1)^{-1/2}}{(-1/2)} + C$$

$$= 2x^2 - \frac{2}{\sqrt{3x^3+1}} + C$$

$$2 = 0 - \frac{2}{1} + C \Rightarrow C = 4$$

$$y = 2x^2 - \frac{2}{\sqrt{3x^3+1}} + 4$$

87. $u = x + 1, x = u - 1, dx = du$

When $x = 0, u = 1$. When $x = 7, u = 8$.

$$\begin{aligned} \text{Area} &= \int_0^7 x \sqrt[3]{x+1} dx = \int_1^8 (u-1) \sqrt[3]{u} du \\ &= \int_1^8 (u^{4/3} - u^{1/3}) du = \left[\frac{3}{7} u^{7/3} - \frac{3}{4} u^{4/3} \right]_1^8 = \left(\frac{384}{7} - 12 \right) - \left(\frac{3}{7} - \frac{3}{4} \right) = \frac{1209}{28} \end{aligned}$$

88. $u = x + 2, x = u - 2, dx = du$

When $x = -2, u = 0$. When $x = 6, u = 8$.

$$\text{Area} = \int_{-2}^6 x^2 \sqrt[3]{x+2} dx = \int_0^8 (u-2)^2 \sqrt[3]{u} du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3}) du = \left[\frac{3}{10} u^{10/3} - \frac{12}{7} u^{7/3} + 3u^{4/3} \right]_0^8 = \frac{4752}{35}$$

89. $A = \int_0^\pi (2 \sin x + \sin 2x) dx = -\left[2 \cos x + \frac{1}{2} \cos 2x \right]_0^\pi = 4$

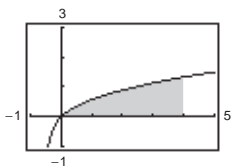
90. $A = \int_0^\pi (\sin x + \cos 2x) dx = \left[-\cos x + \frac{1}{2} \sin 2x \right]_0^\pi = 2$

91. $\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx = 2 \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = \left[2 \tan\left(\frac{x}{2}\right) \right]_{\pi/2}^{2\pi/3} = 2(\sqrt{3} - 1)$

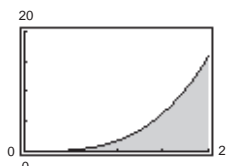
92. Let $u = 2x, du = 2 dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x dx = \frac{1}{2} \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x (2) dx = \left[-\frac{1}{2} \csc 2x \right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

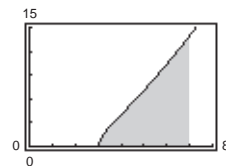
93. $\int_0^4 \frac{x}{\sqrt{2x+1}} dx \approx 3.333 = \frac{10}{3}$



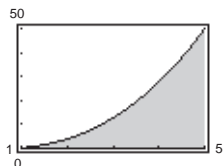
94. $\int_0^2 x^3 \sqrt{x+2} dx \approx 7.581$



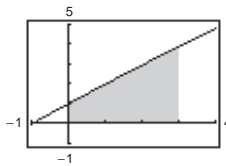
95. $\int_3^7 x \sqrt{x-3} dx \approx 28.8 = \frac{144}{5}$



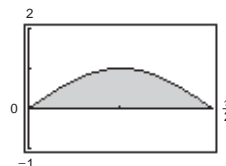
96. $\int_1^5 x^2 \sqrt{x-1} dx \approx 67.505$



97. $\int_0^3 \left(\theta + \cos \frac{\theta}{6} \right) d\theta \approx 7.377$



98. $\int_0^{\pi/2} \sin 2x dx \approx 1.0$



99. $\int (2x-1)^2 dx = \frac{1}{2} \int (2x-1)^2 \cdot 2 dx = \frac{1}{6} (2x-1)^3 + C_1 = \frac{4}{3} x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$$\int (2x-1)^2 dx = \int (4x^2 - 4x + 1) dx = \frac{4}{3} x^3 - 2x^2 + x + C_2$$

They differ by a constant: $C_2 = C_1 - \frac{1}{6}$.

$$\begin{aligned}
 100. \int \sin x \cos x \, dx &= \int (\sin x)^1 (\cos x \, dx) = \frac{\sin^2 x}{2} + C_1 \\
 \int \sin x \cos x \, dx &= - \int (\cos x)^1 (-\sin x \, dx) = -\frac{\cos^2 x}{2} + C_2 \\
 -\frac{\cos^2 x}{2} + C_2 &= -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2
 \end{aligned}$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

101. $f(x) = x^2(x^2 + 1)$ is even.

$$\begin{aligned}
 \int_{-2}^2 x^2(x^2 + 1) \, dx &= 2 \int_0^2 (x^4 + x^2) \, dx = 2 \left[\frac{x^5}{5} + \frac{x^3}{3} \right]_0^2 \\
 &= 2 \left[\frac{32}{5} + \frac{8}{3} \right] = \frac{272}{15}
 \end{aligned}$$

102. $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^2 x(x^2 + 1)^3 \, dx = 0$$

103. $f(x) = \sin^2 x \cos x$ is even.

$$\begin{aligned}
 \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx &= \int_0^{\pi/2} \sin^2 x (\cos x \, dx) \\
 &= 2 \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} \\
 &= \frac{2}{3}
 \end{aligned}$$

104. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$$

105. $\int_0^2 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$; the function x^2 is an even function.

$$(a) \int_{-2}^0 x^2 \, dx = \int_0^2 x^2 \, dx = \frac{8}{3}$$

$$(b) \int_{-2}^2 x^2 \, dx = 2 \int_0^2 x^2 \, dx = \frac{16}{3}$$

$$(c) \int_0^2 (-x^2) \, dx = - \int_0^2 x^2 \, dx = -\frac{8}{3}$$

$$(d) \int_{-2}^0 3x^2 \, dx = 3 \int_0^2 x^2 \, dx = 8$$

106. (a) $\int_{-\pi/4}^{\pi/4} \sin x \, dx = 0$ since $\sin x$ is symmetric to the origin.

$$(b) \int_{-\pi/4}^{\pi/4} \cos x \, dx = 2 \int_0^{\pi/4} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/4} = \sqrt{2} \text{ since } \cos x \text{ is symmetric to the } y\text{-axis.}$$

$$(c) \int_{-\pi/2}^{\pi/2} \cos x \, dx = 2 \int_0^{\pi/2} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/2} = 2$$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$ since $\sin(-x) \cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

$$107. \int_{-4}^4 (x^3 + 6x^2 - 2x - 3) \, dx = \int_{-4}^4 (x^3 - 2x) \, dx + \int_{-4}^4 (6x^2 - 3) \, dx = 0 + 2 \int_0^4 (6x^2 - 3) \, dx = 2 \left[2x^3 - 3x \right]_0^4 = 232$$

$$108. \int_{-\pi}^{\pi} (\sin 3x + \cos 3x) \, dx = \int_{-\pi}^{\pi} \sin 3x \, dx + \int_{-\pi}^{\pi} \cos 3x \, dx = 0 + 2 \int_0^{\pi} \cos 3x \, dx = \left[\frac{2}{3} \sin 3x \right]_0^{\pi} = 0$$

109. If $u = 5 - x^2$, then $du = -2x dx$ and $\int x(5 - x^2)^3 dx = -\frac{1}{2} \int (5 - x^2)^3 (-2x) dx = -\frac{1}{2} \int u^3 du$.

110. $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\int_{-2}^2 x(x^2 + 1)^2 dx = 0$.

111. $\frac{dQ}{dt} = k(100 - t)^2$

$$Q(t) = \int k(100 - t)^2 dt = -\frac{k}{3}(100 - t)^3 + C$$

$$Q(100) = C = 0$$

$$Q(t) = -\frac{k}{3}(100 - t)^3$$

$$Q(0) = -\frac{k}{3}(100)^3 = 2,000,000 \Rightarrow k = -6$$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) = \$250,000$.

112. $\frac{dV}{dt} = \frac{k}{(t + 1)^2}$

$$V(t) = \int \frac{k}{(t + 1)^2} dt = -\frac{k}{t + 1} + C$$

$$V(0) = -k + C = 500,000$$

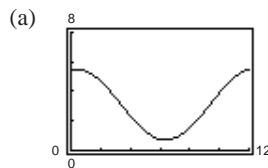
$$V(1) = -\frac{1}{2}k + C = 400,000$$

Solving this system yields $k = -200,000$ and $C = 300,000$. Thus,

$$V(t) = \frac{200,000}{t + 1} + 300,000.$$

When $t = 4$, $V(4) = \$340,000$.

113. $R = 3.121 + 2.399 \sin(0.524t + 1.377)$



Relative minimum: (6.4, 0.7) or June

Relative maximum: (0.4, 5.5) or January

(b) $\int_0^{12} R(t) dt \approx 37.47$ inches

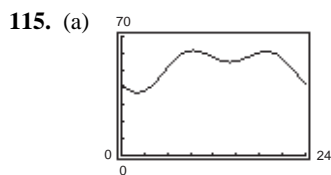
(c) $\frac{1}{3} \int_9^{12} R(t) dt \approx \frac{1}{3}(13) = 4.33$ inches

114. $\frac{1}{b - a} \int_a^b \left[74.50 + 43.75 \sin \frac{\pi t}{6} \right] dt = \frac{1}{b - a} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_a^b$

(a) $\frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^3 = \frac{1}{3} \left(223.5 + \frac{262.5}{\pi} \right) \approx 102.352$ thousand units

(b) $\frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_3^6 = \frac{1}{3} \left(447 + \frac{262.5}{\pi} - 223.5 \right) \approx 102.352$ thousand units

(c) $\frac{1}{12} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^{12} = \frac{1}{12} \left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi} \right) = 74.5$ thousand units



Maximum flow: $R \approx 61.713$ at $t = 9.36$.

[(18.861, 61.178) is a relative maximum.]

(b) Volume = $\int_0^{24} R(t) dt \approx 1272$ (5 thousands of gallons)

$$116. \frac{1}{b-a} \int_a^b [2 \sin(60\pi t) + \cos(120\pi t)] dt = \frac{1}{b-a} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_a^b$$

(a) $\frac{1}{(1/60) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/60} = 60 \left[\left(\frac{1}{30\pi} + 0 \right) - \left(-\frac{1}{30\pi} \right) \right] = \frac{4}{\pi} \approx 1.273$ amps

(b) $\frac{1}{(1/240) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/240} = 240 \left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi} \right) - \left(-\frac{1}{30\pi} \right) \right]$
 $= \frac{2}{\pi} (5 - 2\sqrt{2}) \approx 1.382$ amps

(c) $\frac{1}{(1/30) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/30} = 30 \left[\left(\frac{1}{30\pi} \right) - \left(-\frac{1}{30\pi} \right) \right] = 0$ amps

$$117. u = 1 - x, x = 1 - u, dx = -du$$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{15}{4} x \sqrt{1-x} dx = \frac{15}{4} \int_{1-a}^{1-b} -(1-u) \sqrt{u} du$$

$$= \frac{15}{4} \int_{1-a}^{1-b} (u^{3/2} - u^{1/2}) du = \frac{15}{4} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_{1-a}^{1-b} = \frac{15}{4} \left[\frac{2u^{3/2}}{15} (3u - 5) \right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_a^b$$

(a) $P_{0.50, 0.75} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $P_{0,b} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_0^b = -\frac{(1-b)^{3/2}}{2} (3b+2) + 1 = 0.5$
 $(1-b)^{3/2} (3b+2) = 1$
 $b \approx 0.586 = 58.6\%$

$$118. u = 1 - x, x = 1 - u, dx = -du$$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32} x^3 (1-x)^{3/2} dx = \frac{1155}{32} \int_{1-a}^{1-b} -(1-u)^3 u^{3/2} du$$

$$= \frac{1155}{32} \int_{1-a}^{1-b} (u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2}) du = \frac{1155}{32} \left[\frac{2}{11} u^{11/2} - \frac{2}{3} u^{9/2} + \frac{6}{7} u^{7/2} - \frac{2}{5} u^{5/2} \right]_{1-a}^{1-b}$$

$$= \frac{1155}{32} \left[\frac{2u^{5/2}}{1155} (105u^3 - 385u^2 + 495u - 231) \right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_{1-a}^{1-b}$$

(a) $P_{0, 0.25} = \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_1^{0.75} \approx 0.025 = 2.5\%$

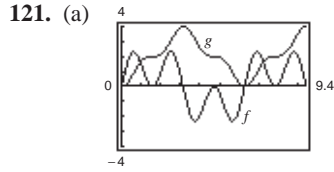
(b) $P_{0.5, 1} = \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_{0.5}^0 \approx 0.736 = 73.6\%$

$$119. (a) C = 0.1 \int_8^{20} \left[12 \sin \frac{\pi(t-8)}{12} \right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t-8)}{12} \right]_8^{20} = \frac{-14.4}{\pi} (-1 - 1) \approx \$9.17$$

(b) $C = 0.1 \int_{10}^{18} \left[12 \sin \frac{\pi(t-8)}{12} - 6 \right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t-8)}{12} - 0.6t \right]_{10}^{18}$
 $= \left[-\frac{14.4}{\pi} \left(\frac{-\sqrt{3}}{2} \right) - 10.8 \right] - \left[-\frac{14.4}{\pi} \left(\frac{\sqrt{3}}{2} \right) - 6 \right] \approx \3.14

Savings $\approx 9.17 - 3.14 = \$6.03$.

120. $\frac{1}{365} \int_0^{365} 100,000 \left[1 + \sin \frac{2\pi(t-60)}{365} \right] dt = \frac{100,000}{365} \left[t - \frac{365}{2\pi} \cos \frac{2\pi(t-60)}{365} \right]_0^{365} = 100,000$ lbs.

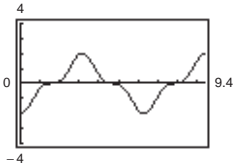


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g .

(d) No, some zeros of f , like $x = \pi/2$, do not correspond to an extrema of g . The graph of g continues to increase after $x = \pi/2$ because f remains above the x -axis.

(e) The graph of h is that of g shifted 2 units downward.



$$g(t) = \int_0^t f(x) dx$$

$$= \int_0^{\pi/2} f(x) dx + \int_{\pi/2}^t f(x) dx = 2 + h(t).$$

122. Let $f(x) = \sin \pi x$, $0 \leq x \leq 1$.

Let $\Delta x = \frac{1}{n}$ and use righthand endpoints

$$c_i = \frac{i}{n}, i = 1, 2, \dots, n.$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\sin(i\pi/n)}{n} = \lim_{\|\Delta x\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x$$

$$= \int_0^1 \sin \pi x dx$$

$$= -\frac{1}{\pi} \cos \pi x \Big|_0^1$$

$$= -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$$

123. (a) Let $u = 1 - x$, $du = -dx$, $x = 1 - u$

$$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 0$$

$$\int_0^1 x^2(1-x)^5 dx = \int_1^0 (1-u)^2 u^5 (-du)$$

$$= \int_0^1 u^5(1-u)^2 du$$

$$= \int_0^1 x^5(1-x)^2 dx$$

(b) Let $u = 1 - x$, $du = -dx$, $x = 1 - u$

$$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 0$$

$$\int_0^1 x^a(1-x)^b dx = \int_1^0 (1-u)^a u^b (-du)$$

$$= \int_0^1 u^b(1-u)^a du$$

$$= \int_0^1 x^b(1-x)^a dx$$

124. (a) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$\text{Let } u = \frac{\pi}{2} - x, du = -dx, x = \frac{\pi}{2} - u:$$

$$\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx = \int_{\pi/2}^0 \cos^2 u (-du)$$

$$= \int_0^{\pi/2} \cos^2 u du = \int_0^{\pi/2} \cos^2 x dx$$

(b) Let $u = \frac{\pi}{2} - x$ as in part (a):

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n\left(\frac{\pi}{2} - x\right) dx = \int_{\pi/2}^0 \cos^n u (-du)$$

$$= \int_0^{\pi/2} \cos^n u du = \int_0^{\pi/2} \cos^n x dx$$

125. False

$$\int (2x + 1)^2 dx = \frac{1}{2} \int (2x + 1)^2 \cdot 2 dx = \frac{1}{6} (2x + 1)^3 + C$$

126. False

$$\int x(x^2 + 1)^2 dx = \frac{1}{2} \int (x^2 + 1)(2x) dx = \frac{1}{4} (x^2 + 1)^2 + C$$

127. True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d) dx = \int_{-10}^{10} (ax^3 + cx) dx + \int_{-10}^{10} (bx^2 + d) dx = 0 + 2 \int_0^{10} (bx^2 + d) dx$$

Odd Even

128. True

$$\int_a^b \sin x dx = \left[-\cos x \right]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi} \sin x dx$$

129. True

$$4 \int \sin x \cos x dx = 2 \int \sin 2x dx = -\cos 2x + C$$

130. False

$$\int \sin^2 2x \cos 2x dx = \frac{1}{2} \int (\sin 2x)^2 (2 \cos 2x) dx = \frac{1}{2} \frac{(\sin 2x)^3}{3} + C = \frac{1}{6} \sin^3 2x + C$$

131. Let $u = cx$, $du = c dx$:

$$\begin{aligned} c \int_a^b f(cx) dx &= c \int_{ca}^{cb} f(u) \frac{du}{c} \\ &= \int_{ca}^{cb} f(u) du \\ &= \int_{ca}^{cb} f(x) dx \end{aligned}$$

132. (a) $\frac{d}{du}[\sin u - u \cos u + C] = \cos u - \cos u + u \sin u = u \sin u$

$$\text{Thus, } \int u \sin u du = \sin u - u \cos u + C.$$

(b) Let $u = \sqrt{x}$, $u^2 = x$, $2u du = dx$.

$$\begin{aligned} \int_0^{\pi^2} \sin \sqrt{x} dx &= \int_0^{\pi} \sin u (2u du) \\ &= 2 \int_0^{\pi} u \sin u du \\ &= 2 \left[\sin u - u \cos u \right]_0^{\pi} \quad (\text{part (a)}) \\ &= 2[-\pi \cos(\pi)] \\ &= 2\pi \end{aligned}$$

133. Because f is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned}\int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= -\int_0^{-a} f(x) dx + \int_0^a f(x) dx.\end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned}\int_{-a}^a f(x) dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x) dx \\ &= -\int_0^a f(u) du + \int_0^a f(x) dx = 0\end{aligned}$$

134. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x + h) dx = \int_{a+h}^{b+h} f(u) du = \int_{a+h}^{b+h} f(x) dx.$$

135. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

$$\begin{aligned}\int_0^1 f(x) dx &= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1} \right]_0^1 \\ &= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0 \quad (\text{Given})\end{aligned}$$

By the Mean Value Theorem for Integrals, there exists c in $[0, 1]$ such that

$$\begin{aligned}\int_0^1 f(x) dx &= f(c)(1 - 0) \\ 0 &= f(c).\end{aligned}$$

Thus the equation has at least one real zero.

$$136. \alpha^2 \int_0^1 f(x) dx = \alpha^2(1) = \alpha^2$$

$$-2\alpha \int_0^1 f(x)x dx = -2\alpha(\alpha) = -2\alpha^2$$

$$\int_0^1 f(x)x^2 dx = \alpha^2$$

Adding,

$$\int_0^1 [\alpha^2 f(x) - 2\alpha x f(x) + x^2 f(x)] dx = 0$$

$$\int_0^1 f(x)(\alpha - x)^2 dx = 0.$$

Since $(\alpha - x)^2 \geq 0$, $f = 0$. Hence, there are no such functions.

Section 4.6 Numerical Integration

1. Exact: $\int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2 dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2 dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{8}{3} \approx 2.6667$

2. Exact: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx = \left[\frac{x^3}{6} + x \right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx \approx \frac{1}{8} \left[1 + 2\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 2\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right) \right] = \frac{75}{64} \approx 1.1719$

Simpson's: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx \approx \frac{1}{12} \left[1 + 4\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 4\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right) \right] = \frac{7}{6} \approx 1.1667$

3. Exact: $\int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{24}{6} = 4.0000$

4. Exact: $\int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = 0.5000$

Trapezoidal: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{8} \left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \frac{1}{4} \right] \approx 0.5090$

Simpson's: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{12} \left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \frac{1}{4} \right] \approx 0.5004$

5. Exact: $\int_0^2 x^3 dx = \left[\frac{1}{4}x^4 \right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{8} \left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0625$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{12} \left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0000$

6. Exact: $\int_0^8 \sqrt[3]{x} dx = \left[\frac{3}{4}x^{4/3} \right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{2} [0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{3} [0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

7. Exact: $\int_4^9 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x} dx \approx \frac{5}{16} \left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3 \right]$
 ≈ 12.6640

Simpson's: $\int_4^9 \sqrt{x} dx \approx \frac{5}{24} \left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3 \right] \approx 12.6667$

8. Exact: $\int_1^3 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\int_1^3 (4 - x^2) dx \approx \frac{1}{4} \left[3 + 2 \left[4 - \left(\frac{3}{2}\right)^2 \right] + 2(0) + 2 \left[4 - \left(\frac{5}{2}\right)^2 \right] - 5 \right] = -0.7500$

Simpson's: $\int_1^3 (4 - x^2) dx \approx \frac{1}{6} \left[3 + 4 \left(4 - \frac{9}{4} \right) + 0 + 4 \left(4 - \frac{25}{4} \right) - 5 \right] \approx -0.6667$

9. Exact: $\int_1^2 \frac{1}{(x+1)^2} dx = \left[-\frac{1}{x+1} \right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$
- Trapezoidal: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{8} \left[\frac{1}{4} + 2 \left(\frac{1}{((5/4)+1)^2} \right) + 2 \left(\frac{1}{((3/2)+1)^2} \right) + 2 \left(\frac{1}{((7/4)+1)^2} \right) + \frac{1}{9} \right]$
 $= \frac{1}{8} \left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9} \right) \approx 0.1676$
- Simpson's: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{12} \left[\frac{1}{4} + 4 \left(\frac{1}{((5/4)+1)^2} \right) + 2 \left(\frac{1}{((3/2)+1)^2} \right) + 4 \left(\frac{1}{((7/4)+1)^2} \right) + \frac{1}{9} \right]$
 $= \frac{1}{12} \left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9} \right) \approx 0.1667$
10. Exact: $\int_0^2 x\sqrt{x^2+1} dx = \frac{1}{3} \left[(x^2+1)^{3/2} \right]_0^2 = \frac{1}{3} (5^{3/2} - 1) \approx 3.393$
- Trapezoidal: $\int_0^2 x\sqrt{x^2+1} dx \approx \frac{1}{4} \left[0 + 2 \left(\frac{1}{2} \right) \sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 2 \left(\frac{3}{2} \right) \sqrt{(3/2)^2+1} + 2\sqrt{2^2+1} \right] \approx 3.457$
- Simpson's: $\int_0^2 x\sqrt{x^2+1} dx \approx \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right) \sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 4 \left(\frac{3}{2} \right) \sqrt{(3/2)^2+1} + 2\sqrt{2^2+1} \right] \approx 3.392$
11. Trapezoidal: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{4} [1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$
- Simpson's: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{6} [1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$
- Graphing utility: 3.241
12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx \approx \frac{1}{4} \left[1 + 2 \left(\frac{1}{\sqrt{1+(1/2)^3}} \right) + 2 \left(\frac{1}{\sqrt{1+1^3}} \right) + 2 \left(\frac{1}{\sqrt{1+(3/2)^3}} \right) + \frac{1}{3} \right] \approx 1.397$
- Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx \approx \frac{1}{6} \left[1 + 4 \left(\frac{1}{\sqrt{1+(1/2)^3}} \right) + 2 \left(\frac{1}{\sqrt{1+1^3}} \right) + 4 \left(\frac{1}{\sqrt{1+(3/2)^3}} \right) + \frac{1}{3} \right] \approx 1.405$
- Graphing utility: 1.402
13. $\int_0^1 \sqrt{x}\sqrt{1-x} dx = \int_0^1 \sqrt{x(1-x)} dx$
- Trapezoidal: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{8} \left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.342$
- Simpson's: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{12} \left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.372$
- Graphing utility: 0.393
14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x dx \approx \frac{\pi}{16} \left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.430$
- Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x dx \approx \frac{\pi}{24} \left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.458$
- Graphing utility: 1.458

15. Trapezoidal:
$$\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{8} \left[\cos 0 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2 \cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$$

$$\approx 0.957$$

Simpson's:
$$\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{12} \left[\cos 0 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4 \cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$$

$$\approx 0.978$$

Graphing utility: 0.977

16. Trapezoidal:
$$\int_0^{\sqrt{\pi/4}} \tan(x^2) dx \approx \frac{\sqrt{\pi/4}}{8} \left[\tan 0 + 2 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$$

$$\approx 0.271$$

Simpson's:
$$\int_0^{\sqrt{\pi/4}} \tan(x^2) dx \approx \frac{\sqrt{\pi/4}}{12} \left[\tan 0 + 4 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$$

$$\approx 0.257$$

Graphing utility: 0.256

17. Trapezoidal:
$$\int_1^{1.1} \sin x^2 dx \approx \frac{1}{80} [\sin(1) + 2 \sin(1.025)^2 + 2 \sin(1.05)^2 + 2 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$$

Simpson's:
$$\int_1^{1.1} \sin x^2 dx \approx \frac{1}{120} [\sin(1) + 4 \sin(1.025)^2 + 2 \sin(1.05)^2 + 4 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$$

Graphing utility: 0.089

18. Trapezoidal:
$$\int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx \approx \frac{\pi}{16} [\sqrt{2} + 2\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 2\sqrt{1 + \cos^2(3\pi/8)} + 1] \approx 1.910$$

Simpson's:
$$\int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx \approx \frac{\pi}{24} [\sqrt{2} + 4\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 4\sqrt{1 + \cos^2(3\pi/8)} + 1] \approx 1.910$$

Graphing utility: 1.910

19. Trapezoidal:
$$\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{32} \left[0 + 2\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.194$$

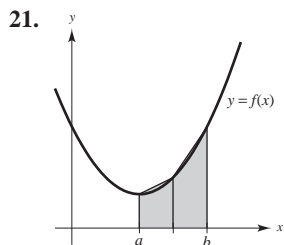
Simpson's:
$$\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{48} \left[0 + 4\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.186$$

Graphing utility: 0.186

20. Trapezoidal:
$$\int_0^{\pi} \frac{\sin x}{x} dx \approx \frac{\pi}{8} \left[1 + \frac{2 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{2 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.836$$

Simpson's:
$$\int_0^{\pi} \frac{\sin x}{x} dx \approx \frac{\pi}{12} \left[1 + \frac{4 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{4 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.852$$

Graphing utility: 1.852



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

23. $f(x) = x^3$
 $f'(x) = 3x^2$
 $f''(x) = 6x$
 $f'''(x) = 6$
 $f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\leq \frac{(2-0)^3}{12(4^2)}(12) = 0.5$ since

$$|f''(x)| \text{ is maximum in } [0, 2] \text{ when } x = 2.$$

(b) Simpson's: Error $\leq \frac{(2-0)^5}{180(4^4)}(0) = 0$ since

$$f^{(4)}(x) = 0.$$

25. $f(x) = \frac{1}{x+1}$
 $f'(x) = \frac{-1}{(x+1)^2}$
 $f''(x) = \frac{2}{(x+1)^3}$
 $f'''(x) = \frac{-6}{(x+1)^4}$
 $f^{(4)}(x) = \frac{24}{(x+1)^5}$

(a) Trapezoidal: Error $\leq \frac{(1-0)^2}{12(4^2)}(2) = \frac{1}{96} \approx 0.01$ since

$$f''(x) \text{ is maximum in } [0, 1] \text{ when } x = 0.$$

(b) Simpson's: Error $\leq \frac{(1-0)^5}{180(4^4)}(24) = \frac{1}{1920} \approx 0.0005$

$$\text{since } f^{(4)}(x) \text{ is maximum in } [0, 1] \text{ when } x = 0.$$

22. Trapezoidal: Linear polynomials
 Simpson's: Quadratic polynomials

24. $f(x) = 2x + 3$
 $f'(x) = 2$
 $f''(x) = 0$

The error is 0 for both rules.

26. $f(x) = (x-1)^{-2}$
 $f'(x) = -2(x-1)^{-3}$
 $f''(x) = 6(x-1)^{-4}$
 $f'''(x) = -24(x-1)^{-5}$
 $f^{(4)}(x) = 120(x-1)^{-6}$

(a) Trapezoidal: Error $\leq \frac{(4-2)^3}{12(4^2)}(6) = \frac{1}{4}$,

$$\text{since } |f''(x)| \text{ is a maximum of } 6 \text{ at } x = 2.$$

(b) Simpson's: Error $\leq \frac{(4-2)^5}{180(4^4)}(120) = \frac{1}{12}$

$$\text{since } |f^{(4)}(x)| \text{ is a maximum of } 120 \text{ at } x = 2.$$

27. $f(x) = \cos x$
 $f'(x) = -\sin x$
 $f''(x) = -\cos x$
 $f'''(x) = \sin x$
 $f^{(4)}(x) = \cos x$

(a) Trapezoidal: Error $\leq \frac{(\pi - 0)^3}{12(4^2)}(1) = \frac{\pi^3}{192} \approx 0.1615$

because $|f''(x)|$ is at most 1 on $[0, \pi]$.

(b) Simpson's: Error $\leq \frac{(\pi - 0)^5}{180(4^4)}(1) = \frac{\pi^5}{46,080} \approx 0.006641$

because $|f^{(4)}(x)|$ is at most 1 on $[0, \pi]$.

28. $f(x) = \sin(\pi x)$
 $f'(x) = \pi \cos(\pi x)$
 $f''(x) = -\pi^2 \sin(\pi x)$
 $f'''(x) = -\pi^3 \cos(\pi x)$
 $f^{(4)}(x) = \pi^4 \sin(\pi x)$

(a) Trapezoidal: Error $\leq \frac{(1 - 0)^3}{12(4^2)}\pi^2 = \frac{\pi^2}{192} \approx 0.0514$,

since $|f''(x)| \leq \pi^2$ on $[0, 1]$.

(b) Simpson's: Error $\leq \frac{(1 - 0)^5}{180(4^4)}\pi^4 = \frac{\pi^4}{46,080} \approx 0.0021$,

since $|f^{(4)}(x)| \leq \pi^4$ on $[0, 1]$.

29. $f''(x) = \frac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\leq \frac{2^3}{12n^2}(2) < 0.00001, n^2 > 133,333.33, n > 365.15$; let $n = 366$.

$$f^{(4)}(x) = \frac{24}{x^5} \text{ in } [1, 3].$$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| = 24$.

Simpson's: Error $\leq \frac{2^5}{180n^4}(24) < 0.00001, n^4 > 426,666.67, n > 25.56$; let $n = 26$.

30. $f''(x) = \frac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \frac{1}{12n^2}(2) < 0.00001, n^2 > 16,666.67, n > 129.10$; let $n = 130$.

$$f^{(4)}(x) = \frac{24}{(1+x)^5} \text{ in } [0, 1]$$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \frac{1}{180n^4}(24) < 0.00001, n^4 > 13,333.33, n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

31. $f(x) = (x + 2)^{1/2}, \quad 0 \leq x \leq 2$

$$f'(x) = \frac{1}{2}(x + 2)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(x + 2)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(x + 2)^{-5/2}$$

$$f^{(4)}(x) = \frac{-15}{16}(x + 2)^{-7/2}$$

(a) Maximum of $|f''(x)| = \left| \frac{-1}{4(x + 2)^{3/2}} \right|$ is $\frac{\sqrt{2}}{16} \approx 0.0884$.

Trapezoidal: Error $\leq \frac{(2 - 0)^3 \left(\frac{\sqrt{2}}{16}\right)}{12n^2} \leq 0.00001$

$$n^2 \geq \frac{8\sqrt{2}}{12(16)} 10^5 = \frac{\sqrt{2}}{24} 10^5$$

$$n \geq 76.8. \text{ Let } n = 77.$$

(b) Maximum of $|f^{(4)}(x)| = \left| \frac{-15}{16(x + 2)^{7/2}} \right|$ is

$$\frac{15\sqrt{2}}{256} \approx 0.0829.$$

Simpson's: Error $\leq \frac{2^5}{180n^4} \left(\frac{15\sqrt{2}}{256}\right) \leq 0.00001$

$$n^4 \geq \frac{32(15)\sqrt{2}}{180(256)} 10^5 = \frac{\sqrt{2}}{96} 10^5$$

$$n \geq 6.2. \text{ Let } n = 8 \text{ (even).}$$

33. $f(x) = \cos(\pi x), \quad 0 \leq x \leq 1$

$$f'(x) = -\pi \sin(\pi x)$$

$$f''(x) = -\pi^2 \cos(\pi x)$$

$$f'''(x) = \pi^3 \sin(\pi x)$$

$$f^{(4)}(x) = \pi^4 \cos(\pi x)$$

(a) Maximum of $|f''(x)| = |-\pi^2 \cos(\pi x)|$ is π^2 .

Trapezoidal: Error $\leq \frac{(1 - 0)^3}{12n^2} \pi^2 \leq 0.00001$

$$n^2 \geq \frac{\pi^2}{12} \cdot 10^5$$

$$n \geq 286.8. \text{ Let } n = 287.$$

(b) Maximum of $|f^{(4)}(x)| = |\pi^4 \cos(\pi x)|$ is π^4 .

Simpson's: Error $\leq \frac{1}{180n^4} \pi^4 \leq 0.00001$

$$n^4 \geq \frac{\pi^4}{180} \cdot 10^5$$

$$n \geq 15.3. \text{ Let } n = 16.$$

32. $f(x) = x^{-1/2}, \quad 1 \leq x \leq 3$

$$f'(x) = \frac{-1}{2}x^{-3/2}$$

$$f''(x) = \frac{3}{4}x^{-5/2}$$

$$f'''(x) = \frac{-15}{8}x^{-7/2}$$

$$f^{(4)}(x) = \frac{105}{16}x^{-9/2}$$

(a) Maximum of $|f''(x)| = \left| \frac{3}{4x^{5/2}} \right|$ is $\frac{3}{4}$ on $[1, 3]$.

Trapezoidal: Error $\leq \frac{(3 - 1)^3 \left(\frac{3}{4}\right)}{12n^2} \leq 0.00001$

$$n^2 \geq \frac{1}{2} \cdot 10^5$$

$$n \geq 223.6. \text{ Let } n = 224$$

(b) Maximum of $|f^{(4)}(x)| = \left| \frac{105}{16x^{9/2}} \right|$ is $\frac{105}{16}$ on $[1, 3]$.

Simpson's: Error $\leq \frac{2^5}{180n^4} \left(\frac{105}{16}\right) \leq 0.00001$

$$n^4 \geq \frac{7}{6} \cdot 10^5$$

$$n \geq 18.5. \text{ Let } n = 20 \text{ (even)}$$

34. $f(x) = \sin x, \quad 0 \leq x \leq \frac{\pi}{2}$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(4)}(x) = \sin x$$

All derivatives are bounded by 1.

(a) Trapezoidal: Error $\leq \frac{(\pi/2)^3}{12n^2} (1) \leq 0.00001$

$$n^2 \geq \frac{\pi^3}{96} \cdot 10^5$$

$$n \geq 179.7. \text{ Let } n = 180.$$

(b) Simpson's: Error $\leq \frac{(\pi/2)^5}{180n^4} (1) \leq 0.00001$

$$n^4 \geq \frac{\pi^5}{5760} \cdot 10^5$$

$$n \geq 8.5. \text{ Let } n = 10 \text{ (even).}$$

35. $f(x) = \sqrt{1+x}$

(a) $f''(x) = -\frac{1}{4(1+x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{1}{4}$.

Trapezoidal: Error $\leq \frac{8}{12n^2} \left(\frac{1}{4}\right) < 0.00001, n^2 > 16,666.67, n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \frac{-15}{16(1+x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{15}{16}$.

Simpson's: Error $\leq \frac{32}{180n^4} \left(\frac{15}{16}\right) < 0.00001, n^4 > 16,666.67, n > 11.36$; let $n = 12$.

36. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\frac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{2}{9}$.

Trapezoidal: Error $\leq \frac{8}{12n^4} \left(\frac{2}{9}\right) < 0.00001, n^2 > 14,814.81, n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\frac{56}{81(x+1)^{10/3}}$ in $[0, 2]$.

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{56}{81}$.

Simpson's: Error $\leq \frac{32}{180n^4} \left(\frac{56}{81}\right) < 0.00001, n^4 > 12,290.81, n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

37. $f(x) = \tan(x^2)$

(a) $f''(x) = 2 \sec^2(x^2)[1 + 4x^2 \tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2} (49.5305) < 0.00001, n^2 > 412,754.17, n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8 \sec^2(x^2)[12x^2 + (3 + 32x^4) \tan(x^2) + 36x^2 \tan^2(x^2) + 48x^4 \tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4} (9184.4734) < 0.00001, n^4 > 5,102,485.22, n > 47.53$; let $n = 48$.

38. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2 \sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2} (2.2853) < 0.00001, n^2 > 19,044.17, n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12) \sin(x^2) - 48x^2 \cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4} (28.4285) < 0.00001, n^4 > 15,793.61, n > 11.21$; let $n = 12$.

39. (a) $b - a = 4 - 0 = 4$, $n = 4$

$$\int_0^4 f(x) dx \approx \frac{4}{8}[3 + 2(7) + 2(9) + 2(7) + 0]$$

$$= \frac{1}{2}(49) = \frac{49}{2} = 24.5$$

$$(b) \int_0^4 f(x) dx \approx \frac{4}{12}[3 + 4(7) + 2(9) + 4(7) + 0]$$

$$= \frac{77}{3} \approx 25.67$$

40. $n = 8$, $b - a = 8 - 0 = 8$

$$(a) \int_0^8 f(x) dx \approx \frac{8}{16}[0 + 2(1.5) + 2(3) + 2(5.5) + 2(9)$$

$$+ 2(10) + 2(9) + 2(6) + 0]$$

$$= \frac{1}{2}(88) = 44$$

$$(b) \int_0^8 f(x) dx \approx \frac{8}{24}[0 + 4(1.5) + 2(3) + 4(5.5) + 2(9)$$

$$+ 4(10) + 2(9) + 4(6) + 0]$$

$$= \frac{1}{3}(134) = \frac{134}{3}$$

41. The program will vary depending upon the computer or programmable calculator that you use.

42. $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

43. $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

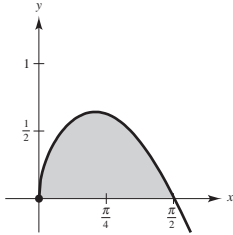
44. $f(x) = \sin \sqrt{x}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

$$45. A = \int_0^{\pi/2} \sqrt{x} \cos x \, dx$$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x} \cos x \, dx \approx \frac{\pi}{84} \left[\sqrt{0} \cos 0 + 4\sqrt{\frac{\pi}{28}} \cos \frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}} \cos \frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}} \cos \frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}} \cos \frac{\pi}{2} \right] \\ \approx 0.701$$



46. Simpson's Rule: $n = 8$

$$8\sqrt{3} \int_0^{\pi/2} \sqrt{1 - \frac{2}{3} \sin^2 \theta} \, d\theta \approx \frac{\sqrt{3}\pi}{6} \left[\sqrt{1 - \frac{2}{3} \sin^2 0} + 4\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{2}} \right] \\ \approx 17.476$$

$$47. W = \int_0^5 100x \sqrt{125 - x^3} \, dx$$

Simpson's Rule: $n = 12$

$$\int_0^5 100x \sqrt{125 - x^3} \, dx \approx \frac{5}{3(12)} \left[0 + 400 \left(\frac{5}{12} \right) \sqrt{125 - \left(\frac{5}{12} \right)^3} + 200 \left(\frac{10}{12} \right) \sqrt{125 - \left(\frac{10}{12} \right)^3} \right. \\ \left. + 400 \left(\frac{15}{12} \right) \sqrt{125 - \left(\frac{15}{12} \right)^3} + \cdots + 0 \right] \approx 10,233.58 \text{ ft} \cdot \text{lb}$$

48. (a) Trapezoidal:

$$\int_0^2 f(x) \, dx \approx \frac{2}{2(8)} [4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x) \, dx \approx \frac{2}{3(8)} [4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844.$$

$$\text{Integrating, } \int_0^2 y \, dx \approx 12.53.$$

$$49. \int_0^{1/2} \frac{6}{\sqrt{1-x^2}} \, dx \quad \text{Simpson's Rule, } n = 6$$

$$\pi \approx \frac{\left(\frac{1}{2} - 0 \right)}{3(6)} [6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282]$$

$$\approx \frac{1}{36} [113.098] \approx 3.1416$$

50. Simpson's Rule: $n = 6$

$$\begin{aligned}\pi &= 4 \int_0^1 \frac{1}{1+x^2} dx \approx \frac{4}{3(6)} \left[1 + \frac{4}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{4}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{4}{1+(5/6)^2} + \frac{1}{2} \right] \\ &\approx 3.14159\end{aligned}$$

51. Area $\approx \frac{1000}{2(10)} [125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89,250$ sq m

52. Area $\approx \frac{120}{2(12)} [75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$
= 7435 sq m

53. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \frac{(b-a)^5}{180n^4}(0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3 dx = \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right)^3 + 1 \right] = \frac{1}{4}$

This is the exact value of the integral.

54. $\int_0^t \sin \sqrt{x} dx = 2, n = 10$

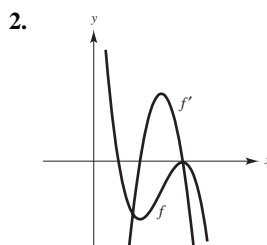
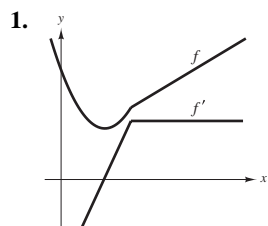
By trial and error, we obtain $t \approx 2.477$.

55. The quadratic polynomial

$$p(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3$$

passes through the three points.

Review Exercises for Chapter 4



3. $\int (2x^2 + x - 1) dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

$$\begin{aligned}4. \int \frac{2}{\sqrt[3]{3x}} dx &= \frac{2}{\sqrt[3]{3}} \int x^{-1/3} dx = \frac{2}{\sqrt[3]{3}} \left(\frac{3}{2} x^{2/3} \right) + C \\ &= \frac{3}{\sqrt[3]{3}} x^{2/3} + C \\ &= (3x)^{2/3} + C\end{aligned}$$

$$5. \int \frac{x^3 + 1}{x^2} dx = \int \left(x + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$$

$$6. \int \frac{x^3 - 2x^2 + 1}{x^2} dx = \int (x - 2 + x^{-2}) dx \\ = \frac{1}{2}x^2 - 2x - \frac{1}{x} + C$$

$$7. \int (4x - 3 \sin x) dx = 2x^2 + 3 \cos x + C$$

$$8. \int (5 \cos x - 2 \sec^2 x) dx = 5 \sin x - 2 \tan x + C$$

$$9. f'(x) = -2x, \quad (-1, 1)$$

$$f(x) = \int -2x dx = -x^2 + C$$

When $x = -1$:

$$y = -1 + C = 1$$

$$C = 2$$

$$y = 2 - x^2$$

$$10. f''(x) = 6(x - 1)$$

$$f'(x) = \int 6(x - 1) dx = 3(x - 1)^2 + C_1$$

Since the slope of the tangent line at $(2, 1)$ is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

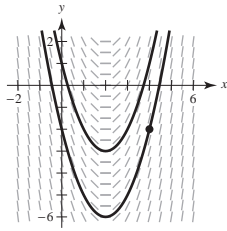
$$f'(x) = 3(x - 1)^2$$

$$f(x) = \int 3(x - 1)^2 dx = (x - 1)^3 + C_2$$

$$f(2) = 1 + C_2 = 1 \text{ when } C_2 = 0.$$

$$f(x) = (x - 1)^3$$

11. (a)

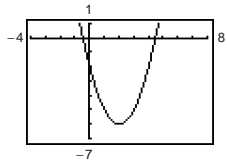


$$(b) \frac{dy}{dx} = 2x - 4, \quad (4, -2)$$

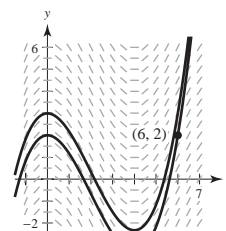
$$y = \int (2x - 4) dx = x^2 - 4x + C$$

$$-2 = 16 - 16 + C \Rightarrow C = -2$$

$$y = x^2 - 4x - 2$$



12. (a)



$$(b) \frac{dy}{dx} = \frac{1}{2}x^2 - 2x, \quad (6, 2)$$

$$y = \int \left(\frac{1}{2}x^2 - 2x \right) dx = \frac{1}{6}x^3 - x^2 + C$$

$$2 = \frac{1}{6}(6^3) - 6^2 + C \Rightarrow C = 2$$

$$y = \frac{1}{6}x^3 - x^2 + 2$$

13. $a(t) = a$

$$v(t) = \int a \, dt = at + C_1$$

$$v(0) = 0 + C_1 = 0 \text{ when } C_1 = 0.$$

$$v(t) = at$$

$$s(t) = \int at \, dt = \frac{a}{2}t^2 + C_2$$

$$s(0) = 0 + C_2 = 0 \text{ when } C_2 = 0.$$

$$s(t) = \frac{a}{2}t^2$$

$$s(30) = \frac{a}{2}(30)^2 = 3600 \text{ or}$$

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$$v(30) = 8(30) = 240 \text{ ft/sec}$$

14. 45 mph = 66 ft/sec

30 mph = 44 ft/sec

$$a(t) = -a$$

$$v(t) = -at + 66 \text{ since } v(0) = 66 \text{ ft/sec.}$$

$$s(t) = -\frac{a}{2}t^2 + 66t \text{ since } s(0) = 0.$$

Solving the system

$$v(t) = -at + 66 = 44$$

$$s(t) = -\frac{a}{2}t^2 + 66t = 264$$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$$475.2 - 264 = 211.2 \text{ ft.}$$

15. $a(t) = -32$

$$v(t) = -32t + 96$$

$$s(t) = -16t^2 + 96t$$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(d) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

16. $a(t) = -9.8 \text{ m/sec}^2$

$$v(t) = -9.8t + v_0 = -9.8t + 40$$

$$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$$

(a) $v(t) = -9.8t + 40 = 0$ when $t = \frac{40}{9.8} \approx 4.08$ sec.

(b) $s(4.08) \approx 81.63$ m

(c) $v(t) = -9.8t + 40 = 20$ when $t = \frac{20}{9.8} \approx 2.04$ sec.

(d) $s(2.04) \approx 61.2$ m

17. $\sum_{i=1}^8 \frac{1}{4i} = \frac{1}{4(1)} + \frac{1}{4(2)} + \cdots + \frac{1}{4(8)}$

18. $\sum_{i=1}^{12} \frac{i+2}{2i} = \frac{1+2}{2(1)} + \frac{2+2}{2(2)} + \frac{3+2}{2(3)} + \cdots + \frac{12+2}{2(12)}$

19. $\sum_{i=1}^n \left(\frac{3}{n}\right)\left(\frac{i+1}{n}\right)^2 = \frac{3}{n}\left(\frac{1+1}{n}\right)^2 + \frac{3}{n}\left(\frac{2+1}{n}\right)^2 + \cdots + \frac{3}{n}\left(\frac{n+1}{n}\right)^2$

20. $\sum_{i=1}^n 3i\left[2 + \frac{(i+1)^2}{n}\right] = 3\left[2 + \frac{4}{n}\right] + 6\left[2 + \frac{9}{n}\right] + \cdots + 3n\left[2 + \frac{(n+1)^2}{n}\right]$

21. $\sum_{i=1}^{10} 3i = 3\left(\frac{10(11)}{2}\right) = 165$

22. $\sum_{i=1}^{20} (4i - 1) = 4\frac{20(21)}{2} - 20 = 820$

23.
$$\begin{aligned} \sum_{i=1}^{20} (i+1)^2 &= \sum_{i=1}^{20} (i^2 + 2i + 1) \\ &= \frac{20(21)(41)}{6} + 2\frac{20(21)}{2} + 20 \\ &= 2870 + 420 + 20 = 3310 \end{aligned}$$

24.
$$\begin{aligned} \sum_{i=1}^{12} i(i^2 - 1) &= \sum_{i=1}^{12} (i^3 - i) \\ &= \frac{(12^2)(13^2)}{4} - \frac{12(13)}{2} \\ &= 6084 - 78 = 6006 \end{aligned}$$

25. (a) $\sum_{i=1}^{10} (2i - 1)$

(b) $\sum_{i=1}^n i^3$

(c) $\sum_{i=1}^{10} (4i + 2)$

26. $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5} \sum_{i=1}^5 x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

(b) $\sum_{i=1}^5 \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

(c) $\sum_{i=1}^5 (2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^5 (x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

27. $y = \frac{10}{x^2 + 1}, \Delta x = \frac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2} \left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1} \right]$$

$$\approx 13.0385$$

$$s(n) = s(4) = \frac{1}{2} \left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1} \right]$$

$$\approx 9.0385$$

$$9.0385 < \text{Area of Region} < 13.0385$$

28. $y = 9 - \frac{1}{4}x^2, \Delta x = 1, n = 4$

$$S(4) = 1 \left[\left(9 - \frac{1}{4}(4)\right) + \left(9 - \frac{1}{4}(9)\right) + \left(9 - \frac{1}{4}(16)\right) + 9 - \frac{1}{4}(25) \right]$$

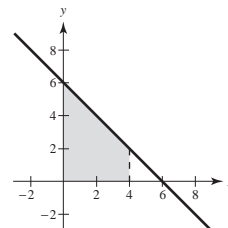
$$\approx 22.5$$

$$s(4) = 1 \left[\left(9 - \frac{1}{4}(9)\right) + \left(9 - \frac{1}{4}(16)\right) + \left(9 - \frac{1}{4}(25)\right) + (9 - 9) \right]$$

$$\approx 14.5$$

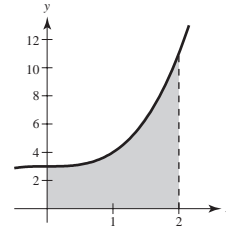
29. $y = 6 - x, \Delta x = \frac{4}{n}, \text{right endpoints}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 - \frac{4i}{n}\right) \frac{4}{n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[6n - \frac{4n(n+1)}{2}\right] \\ &= \lim_{n \rightarrow \infty} \left[24 - 8 \frac{n+1}{n}\right] = 24 - 8 = 16 \end{aligned}$$



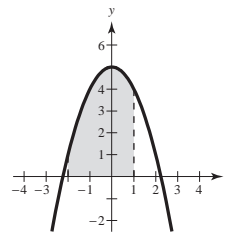
30. $y = x^2 + 3$, $\Delta x = \frac{2}{n}$ right endpoints

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^2 + 3 \right] \left(\frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left[\frac{4i^2}{n^2} + 3 \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} + 3n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \frac{(n+1)(2n+1)}{n^2} + 6 \right] = \frac{8}{3} + 6 = \frac{26}{3} \end{aligned}$$



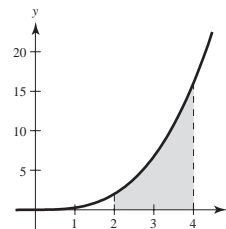
31. $y = 5 - x^2$, $\Delta x = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 - \left(-2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[n + \frac{12}{n} \frac{n(n+1)}{2} - \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[3 + 18 \frac{n+1}{n} - \frac{9}{2} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= 3 + 18 - 9 = 12 \end{aligned}$$



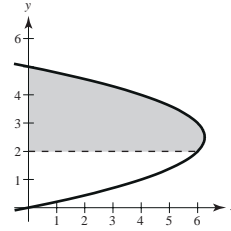
32. $y = \frac{1}{4}x^3$, $\Delta x = \frac{2}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{4} \left(2 + \frac{2i}{n} \right)^3 \left(\frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=1}^n \left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[n + \frac{3}{n} \frac{n(n+1)}{2} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3} \frac{n^2(n+1)^2}{4} \right] \\ &= 4 + 6 + 4 + 1 = 15 \end{aligned}$$



$$33. x = 5y - y^2, 2 \leq y \leq 5, \Delta y = \frac{3}{n}$$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 \left(2 + \frac{3i}{n} \right) - \left(2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[10 + \frac{15i}{n} - 4 - 12 \frac{i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[6n + \frac{3n(n+1)}{2} - \frac{9n(n+1)(2n+1)}{6} \right] \\ &= \left[18 + \frac{9}{2} - 9 \right] = \frac{27}{2} \end{aligned}$$



$$34. (a) S = m \left(\frac{b}{4} \right) \left(\frac{b}{4} \right) + m \left(\frac{2b}{4} \right) \left(\frac{b}{4} \right) + m \left(\frac{3b}{4} \right) \left(\frac{b}{4} \right) + m \left(\frac{4b}{4} \right) \left(\frac{b}{4} \right) = \frac{mb^2}{16} (1 + 2 + 3 + 4) = \frac{5mb^2}{8}$$

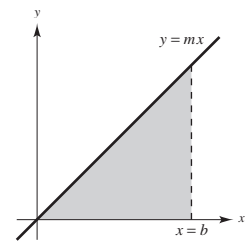
$$s = m(0) \left(\frac{b}{4} \right) + m \left(\frac{b}{4} \right) \left(\frac{b}{4} \right) + m \left(\frac{2b}{4} \right) \left(\frac{b}{4} \right) + m \left(\frac{3b}{4} \right) \left(\frac{b}{4} \right) = \frac{mb^2}{16} (1 + 2 + 3) = \frac{3mb^2}{8}$$

$$(b) S(n) = \sum_{i=1}^n f \left(\frac{bi}{n} \right) \left(\frac{b}{n} \right) = \sum_{i=1}^n \left(\frac{mbi}{n} \right) \left(\frac{b}{n} \right) = m \left(\frac{b}{n} \right)^2 \sum_{i=1}^n i = \frac{mb^2}{n^2} \left(\frac{n(n+1)}{2} \right) = \frac{mb^2(n+1)}{2n}$$

$$s(n) = \sum_{i=0}^{n-1} f \left(\frac{bi}{n} \right) \left(\frac{b}{n} \right) = \sum_{i=0}^{n-1} m \left(\frac{bi}{n} \right) \left(\frac{b}{n} \right) = m \left(\frac{b}{n} \right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2} \left(\frac{(n-1)n}{2} \right) = \frac{mb^2(n-1)}{2n}$$

$$(c) \text{Area} = \lim_{n \rightarrow \infty} \frac{mb^2(n+1)}{2n} = \lim_{n \rightarrow \infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2} mb^2 = \frac{1}{2} (b)(mb) = \frac{1}{2} (\text{base})(\text{height})$$

$$(d) \int_0^b mx \, dx = \left[\frac{1}{2} mx^2 \right]_0^b = \frac{1}{2} mb^2$$



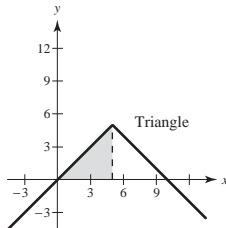
$$35. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (2c_i - 3) \Delta x_i = \int_4^6 (2x - 3) \, dx$$

$$36. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 3c_i(9 - c_i^2) \Delta x_i = \int_1^3 3x(9 - x^2) \, dx$$

$$37. \int_{-2}^0 (3x + 6) \, dx$$

$$38. \int_{-3}^3 (9 - x^2) \, dx$$

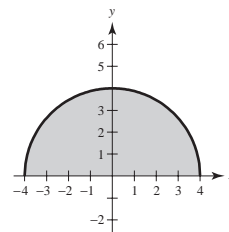
39.



$$\int_0^5 (5 - |x - 5|) \, dx = \int_0^5 (5 - (5 - x)) \, dx = \int_0^5 x \, dx = \frac{25}{2}$$

(triangle)

40.



$$\int_{-4}^4 \sqrt{16 - x^2} \, dx = \frac{1}{2} \pi (4)^2 = 8\pi$$

(semicircle)

$$41. (a) \int_2^6 [f(x) + g(x)] dx = \int_2^6 f(x) dx + \int_2^6 g(x) dx = 10 + 3 = 13$$

$$(b) \int_2^6 [f(x) - g(x)] dx = \int_2^6 f(x) dx - \int_2^6 g(x) dx = 10 - 3 = 7$$

$$(c) \int_2^6 [2f(x) - 3g(x)] dx = 2 \int_2^6 f(x) dx - 3 \int_2^6 g(x) dx = 2(10) - 3(3) = 11$$

$$(d) \int_2^6 5f(x) dx = 5 \int_2^6 f(x) dx = 5(10) = 50$$

$$42. (a) \int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 4 + (-1) = 3$$

$$(b) \int_6^3 f(x) dx = - \int_3^6 f(x) dx = -(-1) = 1$$

$$(c) \int_4^4 f(x) dx = 0$$

$$(d) \int_3^6 -10f(x) dx = -10 \int_3^6 f(x) dx = -10(-1) = 10$$

$$43. \int_0^4 (2 + x) dx = \left[2x + \frac{x^2}{2} \right]_0^4 = 8 + \frac{16}{2} = 16$$

$$44. \int_{-1}^1 (t^2 + 2) dt = \left[\frac{t^3}{3} + 2t \right]_{-1}^1 = \frac{14}{3}$$

$$45. \int_{-1}^1 (4t^3 - 2t) dt = \left[t^4 - t^2 \right]_{-1}^1 = 0$$

$$46. \int_{-2}^2 (x^4 + 2x^2 - 5) dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} - 5x \right]_{-2}^2$$

$$= \left(\frac{32}{5} + \frac{16}{3} - 10 \right) - \left(-\frac{32}{5} - \frac{16}{3} + 10 \right)$$

$$= \frac{52}{15}$$

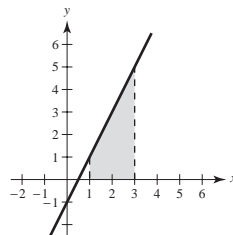
$$47. \int_4^9 x\sqrt{x} dx = \int_4^9 x^{3/2} dx = \left[\frac{2}{5} x^{5/2} \right]_4^9 = \frac{2}{5} [(\sqrt{9})^5 - (\sqrt{4})^5] = \frac{2}{5} (243 - 32) = \frac{422}{5}$$

$$48. \int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3} \right) dx = \int_1^2 (x^{-2} - x^{-3}) dx = \left[-\frac{1}{x} + \frac{1}{2x^2} \right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8} \right) - \left(-1 + \frac{1}{2} \right) = \frac{1}{8}$$

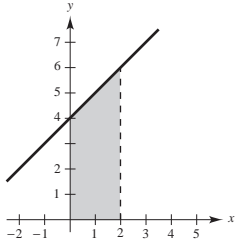
$$49. \int_0^{3\pi/4} \sin \theta d\theta = \left[-\cos \theta \right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2} \right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$$

$$50. \int_{-\pi/4}^{\pi/4} \sec^2 t dt = \left[\tan t \right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$$

$$51. \int_1^3 (2x - 1) dx = \left[x^2 - x \right]_1^3 = 6$$



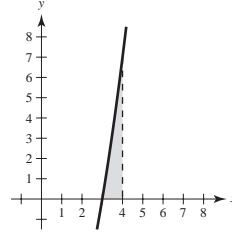
$$52. \int_0^2 (x + 4) dx = \left[\frac{x^2}{2} + 4x \right]_0^2 = 10$$



$$53. \int_3^4 (x^2 - 9) dx = \left[\frac{x^3}{3} - 9x \right]_3^4$$

$$= \left(\frac{64}{3} - 36 \right) - (9 - 27)$$

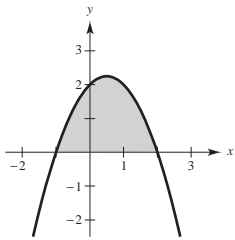
$$= \frac{64}{3} - \frac{54}{3} = \frac{10}{3}$$



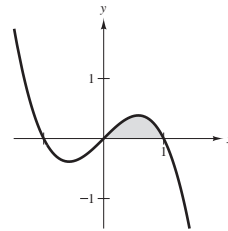
$$54. \int_{-1}^2 (-x^2 + x + 2) dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^2$$

$$= \left(-\frac{8}{3} + 2 + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right)$$

$$= \frac{10}{3} + \frac{7}{6} = \frac{9}{2}$$



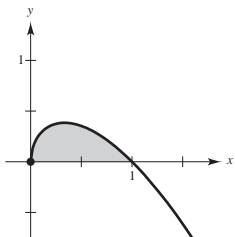
$$55. \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$



$$56. \int_0^1 \sqrt{x}(1-x) dx = \int_0^1 (x^{1/2} - x^{3/2}) dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^1$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$



$$57. \text{Area} = \int_0^1 \sin x dx$$

$$= -\cos x \Big|_0^1$$

$$= -\cos(1) + 1$$

$$= 1 - \cos(1)$$

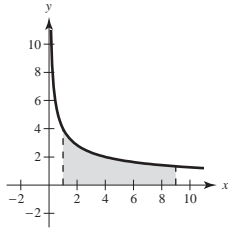
$$\approx 0.460$$

$$58. \text{Area} = \int_0^{\pi/2} (x + \cos x) dx$$

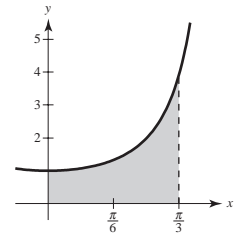
$$= \left[\frac{x^2}{2} + \sin x \right]_0^{\pi/2}$$

$$= \frac{\pi^2}{8} + 1$$

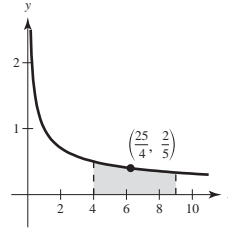
59. Area = $\int_1^9 \frac{4}{\sqrt{x}} dx = \left[\frac{4x^{1/2}}{(1/2)} \right]_1^9 = 8(3 - 1) = 16$



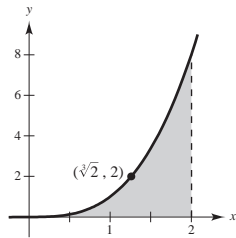
60. Area = $\int_0^{\pi/3} \sec^2 x dx$
 $= \tan x \Big|_0^{\pi/3} = \sqrt{3}$



61. $\frac{1}{9 - 4} \int_4^9 \frac{1}{\sqrt{x}} dx = \left[\frac{1}{5} 2\sqrt{x} \right]_4^9 = \frac{2}{5}(3 - 2) = \frac{2}{5}$ Average value
 $\frac{2}{5} = \frac{1}{\sqrt{x}}$
 $\sqrt{x} = \frac{5}{2}$
 $x = \frac{25}{4}$



62. $\frac{1}{2 - 0} \int_0^2 x^3 dx = \left[\frac{x^4}{8} \right]_0^2 = 2$
 $x^3 = 2$
 $x = \sqrt[3]{2}$



63. $F'(x) = x^2 \sqrt{1 + x^3}$

64. $F'(x) = \frac{1}{x^2}$

65. $F'(x) = x^2 + 3x + 2$

66. $F'(x) = \csc^2 x$

67. $\int (x^2 + 1)^3 dx = \int (x^6 + 3x^4 + 3x^2 + 1) dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$

68. $\int \left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2}) dx = \frac{x^3}{3} + 2x - \frac{1}{x} + C$

69. $u = x^3 + 3, du = 3x^2 dx$

$\int \frac{x^2}{\sqrt{x^3 + 3}} dx = \int (x^3 + 3)^{-1/2} x^2 dx = \frac{1}{3} \int (x^3 + 3)^{-1/2} 3x^2 dx = \frac{2}{3}(x^3 + 3)^{1/2} + C$

70. $u = x^3 + 3, du = 3x^2 dx$

$\int x^2 \sqrt{x^3 + 3} dx = \frac{1}{3} \int (x^3 + 3)^{1/2} 3x^2 dx = \frac{2}{9}(x^3 + 3)^{3/2} + C$

71. $u = 1 - 3x^2, du = -6x dx$

$\int x(1 - 3x^2)^4 dx = -\frac{1}{6} \int (1 - 3x^2)^4 (-6x dx) = -\frac{1}{30}(1 - 3x^2)^5 + C = \frac{1}{30}(3x^2 - 1)^5 + C$

72. $u = x^2 + 6x - 5, du = (2x + 6) dx$

$$\int \frac{x + 3}{(x^2 + 6x - 5)^2} dx = \frac{1}{2} \int \frac{2x + 6}{(x^2 + 6x - 5)^2} dx = \frac{-1}{2(x^2 + 6x - 5)} + C = \frac{-1}{2(x^2 + 6x - 5)} + C$$

73. $\int \sin^3 x \cos x dx = \frac{1}{4} \sin^4 x + C$

74. $\int x \sin 3x^2 dx = \frac{1}{6} \int (\sin 3x^2)(6x) dx = -\frac{1}{6} \cos 3x^2 + C$

75. $\int \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta = \int (1 - \cos \theta)^{-1/2} \sin \theta d\theta = 2(1 - \cos \theta)^{1/2} + C = 2\sqrt{1 - \cos \theta} + C$

76. $\int \frac{\cos x}{\sqrt{\sin x}} dx = \int (\sin x)^{-1/2} \cos x dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$

77. $\int \tan^n x \sec^2 x dx = \frac{\tan^{n+1} x}{n+1} + C, n \neq -1$

78. $\int \sec 2x \tan 2x dx = \frac{1}{2} \int (\sec 2x \tan 2x)(2) dx = \frac{1}{2} \sec 2x + C$

79. $\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x dx = \frac{1}{\pi} \int (1 + \sec \pi x)^2 (\pi \sec \pi x \tan \pi x) dx = \frac{1}{3\pi} (1 + \sec \pi x)^3 + C$

80. $\int \cot^4 \alpha \csc^2 \alpha d\alpha = -\int (\cot \alpha)^4 (-\csc^2 \alpha) d\alpha = -\frac{1}{5} \cot^5 \alpha + C$

81. $\int_{-1}^2 x(x^2 - 4) dx = \frac{1}{2} \int_{-1}^2 (x^2 - 4)(2x) dx = \frac{1}{2} \left[\frac{(x^2 - 4)^2}{2} \right]_{-1}^2 = \frac{1}{4} [0 - 9] = -\frac{9}{4}$

82. $\int_0^1 x^2(x^3 + 1)^3 dx = \frac{1}{3} \int_0^1 (x^3 + 1)^3 (3x^2) dx = \frac{1}{12} [(x^3 + 1)^4]_0^1 = \frac{1}{12} (16 - 1) = \frac{5}{4}$

83. $\int_0^3 \frac{1}{\sqrt{1+x}} dx = \int_0^3 (1+x)^{-1/2} dx = \left[2(1+x)^{1/2} \right]_0^3 = 4 - 2 = 2$

84. $\int_3^6 \frac{x}{3\sqrt{x^2-8}} dx = \frac{1}{6} \int_3^6 (x^2-8)^{-1/2} (2x) dx = \left[\frac{1}{3} (x^2-8)^{1/2} \right]_3^6 = \frac{1}{3} (2\sqrt{7} - 1)$

85. $u = 1 - y, y = 1 - u, dy = -du$

When $y = 0, u = 1$. When $y = 1, u = 0$.

$$\begin{aligned} 2\pi \int_0^1 (y+1)\sqrt{1-y} dy &= 2\pi \int_1^0 -[(1-u)+1]\sqrt{u} du \\ &= 2\pi \int_1^0 (u^{3/2} - 2u^{1/2}) du = 2\pi \left[\frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} \right]_1^0 = \frac{28\pi}{15} \end{aligned}$$

86. $u = x + 1, x = u - 1, dx = du$

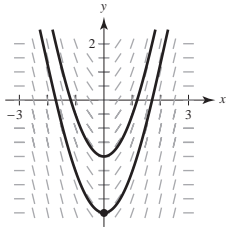
When $x = -1, u = 0$. When $x = 0, u = 1$.

$$\begin{aligned} 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx &= 2\pi \int_0^1 (u-1)^2 \sqrt{u} du \\ &= 2\pi \int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2}) du = 2\pi \left[\frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_0^1 = \frac{32\pi}{105} \end{aligned}$$

87. $\int_0^\pi \cos\left(\frac{x}{2}\right) dx = 2 \int_0^\pi \cos\left(\frac{x}{2}\right) \frac{1}{2} dx = \left[2 \sin\left(\frac{x}{2}\right) \right]_0^\pi = 2$

88. $\int_{-\pi/4}^{\pi/4} \sin 2x dx = 0$ since $\sin 2x$ is an odd function.

89. (a)

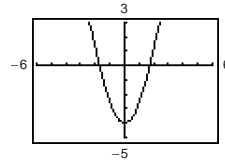


(b) $\frac{dy}{dx} = x\sqrt{9-x^2}, \quad (0, -4)$

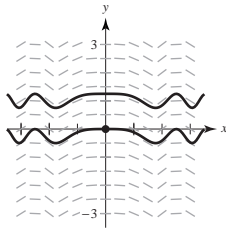
$$y = \int (9-x^2)^{1/2} x dx = \frac{-1}{2} \frac{(9-x^2)^{3/2}}{3/2} + C = -\frac{1}{3}(9-x^2)^{3/2} + C$$

$$-4 = -\frac{1}{3}(9-0)^{3/2} + C = -\frac{1}{3}(27) + C \Rightarrow C = 5$$

$$y = -\frac{1}{3}(9-x^2)^{3/2} + 5$$



90. (a)



(b) $\frac{dy}{dx} = -\frac{1}{2}x \sin(x^2), \quad (0, 0)$

$$y = \int -\frac{1}{2}x \sin(x^2) dx = -\frac{1}{4} \int \sin(x^2)(2x dx) \quad (u = x^2)$$

$$= -\frac{1}{4}(-\cos(x^2)) + C$$

$$= \frac{1}{4}\cos(x^2) + C$$

$$0 = \frac{1}{4}\cos(0) + C \Rightarrow C = -\frac{1}{4}$$

$$y = \frac{1}{4}\cos(x^2) - \frac{1}{4}$$

91. $\int_1^9 x(x-1)^{1/3} dx$. Let $u = x-1, du = dx$.

$$\begin{aligned} A &= \int_0^8 (u+1)u^{1/3} du = \int_0^8 (u^{4/3} + u^{1/3}) du \\ &= \left[\frac{3u^{7/3}}{7} + \frac{3u^{4/3}}{4} \right]_0^8 \\ &= \frac{3}{7}(128) + \frac{3}{4}(16) = \frac{468}{7} \end{aligned}$$

92. $\int_0^{\pi/2} (\cos x + \sin(2x)) dx = \left[\sin x - \frac{1}{2} \cos(2x) \right]_0^{\pi/2}$
 $= \left(1 + \frac{1}{2} \right) - \left(0 - \frac{1}{2} \right) = 2$

93. $p = 1.20 + 0.04t$

$$C = \frac{15,000}{M} \int_t^{t+1} p ds = \frac{15,000}{M} \int_t^{t+1} (1.20 + 0.04s) ds$$

(a) 2000 corresponds to $t = 10$.

$$\begin{aligned} C &= \frac{15,000}{M} \int_{10}^{11} [1.20 + 0.04t] dt \\ &= \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{10}^{11} = \frac{24,300}{M} \end{aligned}$$

(b) 2005 corresponds to $t = 15$.

$$C = \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{15}^{16} = \frac{27,300}{M}$$

$$94. \int_0^2 1.75 \sin \frac{\pi t}{2} dt = -\frac{2}{\pi} \left[1.75 \cos \frac{\pi t}{2} \right] = -\frac{2}{\pi} (1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282 \text{ liters}$$

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

$$95. \text{ Trapezoidal Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{8} \left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.257$$

$$\text{Simpson's Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{12} \left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.254$$

Graphing utility: 0.254

$$96. \text{ Trapezoidal Rule } (n = 4): \int_0^1 \frac{x^{3/2}}{3-x^2} dx \approx \frac{1}{8} \left[0 + \frac{2(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{2(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2} \right] \approx 0.172$$

$$\text{Simpson's Rule } (n = 4): \int_0^1 \frac{x^{3/2}}{3-x^2} dx \approx \frac{1}{12} \left[0 + \frac{4(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{4(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2} \right] \approx 0.166$$

Graphing utility: 0.166

$$97. \text{ Trapezoidal Rule } (n = 4): \int_0^{\pi/2} \sqrt{x} \cos x dx \approx 0.637$$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

$$98. \text{ Trapezoidal Rule } (n = 4): \int_0^{\pi} \sqrt{1 + \sin^2 x} dx \approx 3.820$$

Simpson's Rule $(n = 4)$: 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

$$1. \text{ (a) } L(1) = \int_1^1 \frac{1}{t} dt = 0$$

$$\text{(c) } L(x) = 1 = \int_1^x \frac{1}{t} dt \text{ for } x \approx 2.718$$

$$\int_1^{2.718} \frac{1}{t} dt = 0.999896$$

(Note: The exact value of x is e , the base of the natural logarithm function.)

$$\text{(b) } L'(x) = \frac{1}{x} \text{ by the Second Fundamental Theorem of Calculus.}$$

$$L'(1) = 1$$

$$\text{(d) We first show that } \int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{u} du.$$

$$\text{To see this, let } u = \frac{t}{x_1} \text{ and } du = \frac{1}{x_1} dt.$$

$$\text{Then } \int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{ux_1} (x_1 du) = \int_{1/x_1}^1 \frac{1}{u} du = \int_{1/x_1}^1 \frac{1}{u} du.$$

$$\text{Now, } L(x_1 x_2) = \int_1^{x_1 x_2} \frac{1}{t} dt = \int_{1/x_1}^{x_2} \frac{1}{u} du \text{ (using } u = \frac{t}{x_1} \text{)}$$

$$= \int_{1/x_1}^1 \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du$$

$$= \int_1^{x_1} \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du$$

$$= L(x_1) + L(x_2).$$

2. (a) $F(x) = \int_2^x \sin t^2 dt$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	-0.8048	-0.4945	-0.0265	0.0611	0	-0.0867	-0.3743	-0.0312	-0.0576	-0.2769

(b) $G(x) = \frac{1}{x-2} \int_2^x \sin t^2 dt$

x	1.9	1.95	1.99	2.01	2.1
$G(x)$	-0.6106	-0.6873	-0.7436	-0.7697	-0.8671

$$\lim_{x \rightarrow 2} G(x) \approx -0.75$$

(c) $F'(2) = \lim_{x \rightarrow 2} \frac{F(x) - F(2)}{x - 2}$

$$= \lim_{x \rightarrow 2} \frac{1}{x-2} \int_2^x \sin t^2 dt$$

$$= \lim_{x \rightarrow 2} G(x)$$

Since $F'(x) = \sin x^2$, $F'(2) = \sin 4 = \lim_{x \rightarrow 2} G(x)$. (**Note:** $\sin 4 \approx -0.7568$)

3. $y = x^4 - 4x^3 + 4x^2$, $[0, 2]$, $c_i = \frac{2i}{n}$

(a) $\Delta x = \frac{2}{n}$, $f(x) = x^4 - 4x^3 + 4x^2$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^4 - 4 \left(\frac{2i}{n} \right)^3 + 4 \left(\frac{2i}{n} \right)^2 \right] \frac{2}{n}$$

(b) $\sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^4 - 4 \left(\frac{2i}{n} \right)^3 + 4 \left(\frac{2i}{n} \right)^2 \right] \frac{2}{n}$

$$= \left[\frac{8(n+1)(6n^3 + 9n^2 + n - 1)}{15n^3} - \frac{8(n+1)^2}{n} + \frac{8(n+1)(2n+1)}{3n} \right] \frac{2}{n}$$

$$= \left[\frac{8(n+1)(n^3 - n^2 + n - 1)}{15n^3} \right] \frac{2}{n}$$

(c) $A = \lim_{n \rightarrow \infty} \left[\frac{8(n+1)(n^3 - n^2 + n - 1)}{15n^3} \right] \frac{2}{n} = \frac{16}{15}$

4. $y = \frac{1}{2}x^5 + 2x^3$, $[0, 2]$, $c_i = \frac{2i}{n}$, $\Delta x = \frac{2}{n}$

(a) $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{1}{2} \left(\frac{2i}{n} \right)^5 + 2 \left(\frac{2i}{n} \right)^3 \right] \frac{2}{n}$$

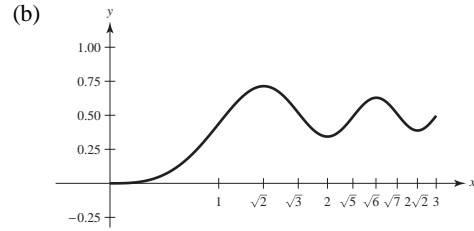
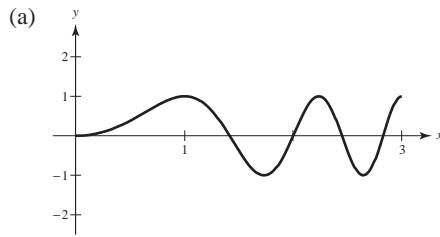
(b) $\sum_{i=1}^n \left[\frac{1}{2} \left(\frac{2i}{n} \right)^5 + 2 \left(\frac{2i}{n} \right)^3 \right] \frac{2}{n}$

$$= \left[\frac{4(n+1)^2(2n^2 + 2n - 1)}{3n^3} + \frac{4(n+1)^2}{n} \right] \frac{2}{n}$$

$$= \left[\frac{8(n+1)^2(5n^2 + 2n - 1)}{3n^4} \right]$$

(c) $A = \lim_{n \rightarrow \infty} \left[\frac{8(n+1)^2(5n^2 + 2n - 1)}{3n^4} \right] = \frac{40}{3}$

5. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$



The zeros of $y = \sin \frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin \frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}, n \text{ integer.}$

Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n}, n \text{ integer}$

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \text{ and } \sqrt{7}$.

6. (a) $\int_{-1}^1 \cos x \, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2 \cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\int_{-1}^1 \cos x \, dx = \sin x \Big|_{-1}^1 = 2 \sin(1) \approx 1.6829$

Error: $|1.6829 - 1.6758| = 0.0071$

(b) $\int_{-1}^1 \frac{1}{1+x^2} \, dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$

(Note: exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$\int_{-1}^1 p(x) \, dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx \right]_{-1}^1 = \frac{2b}{3} + 2d$

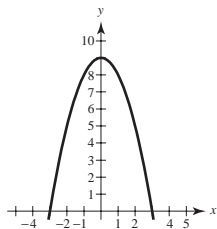
$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$

7. (a) Area = $\int_{-3}^3 (9 - x^2) \, dx = 2 \int_0^3 (9 - x^2) \, dx$

$= 2 \left[9x - \frac{x^3}{3} \right]_0^3$

$= 2[27 - 9] = 36$

(b) Base = 6, height = 9, Area = $\frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$



(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

Area = $2 \int_0^{b/a} (b^2 - a^2x^2) \, dx$

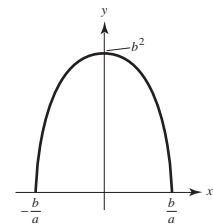
$= 2 \left[b^2x - a^2 \frac{x^3}{3} \right]_0^{b/a}$

$= 2 \left[b^2 \left(\frac{b}{a}\right) - \frac{a^2 \left(\frac{b}{a}\right)^3}{3} \right]$

$= 2 \left[\frac{b^3}{a} - \frac{1}{3} \frac{b^3}{a} \right] = \frac{4}{3} \frac{b^3}{a}$

Base = $\frac{2b}{a}$, height = b^2

Archimedes' Formula: Area = $\frac{2}{3} \left(\frac{2b}{a}\right)(b^2) = \frac{4}{3} \frac{b^3}{a}$



8. Let d be the distance traversed and a be the uniform acceleration. We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$$s(t) = d \text{ when } t = \sqrt{\frac{2d}{a}}.$$

$$\text{The highest speed is } v = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}.$$

The lowest speed is $v = 0$.

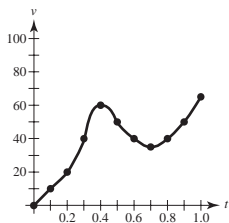
$$\text{The mean speed is } \frac{1}{2}(\sqrt{2ad} + 0) = \sqrt{\frac{ad}{2}}.$$

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

10. (a)



(b) v is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

$$(c) \text{ Average acceleration} = \frac{v(0.4) - v(0)}{0.4 - 0} = \frac{60 - 0}{0.4} = 150 \text{ mi/hr}^2$$

(d) This integral is the total distance traveled in miles.

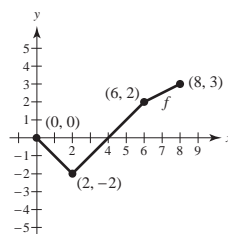
$$\int_0^1 v(t) dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

9. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

$$(c) f(x) = \begin{cases} -x, & 0 \leq x < 2 \\ x - 4, & 2 \leq x < 6 \\ \frac{1}{2}x - 1, & 6 \leq x \leq 8 \end{cases}$$

$$F(x) = \int_0^x f(t) dt = \begin{cases} (-x^2/2), & 0 \leq x < 2 \\ (x^2/2) - 4x + 4, & 2 \leq x < 6 \\ (1/4)x^2 - x - 5, & 6 \leq x \leq 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

$$(d) F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

11. $\int_0^x f(t)(x-t) dt = \int_0^x xf(t) dt - \int_0^x tf(t) dt = x \int_0^x f(t) dt - \int_0^x tf(t) dt$

Thus, $\frac{d}{dx} \int_0^x f(t)(x-t) dt = xf(x) + \int_0^x f(t) dt - xf(x) = \int_0^x f(t) dt$

Differentiating the other integral,

$$\frac{d}{dx} \int_0^x \left(\int_0^x f(v) dv \right) dt = \int_0^x f(v) dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x-t) dt = \int_0^x \left(\int_0^t f(v) dv \right) dt + C.$$

Letting $x = 0$, we see that $C = 0$.

12. Consider $F(x) = [f(x)]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. Thus,

$$\begin{aligned} \int_a^b f(x)f'(x) dx &= \int_a^b \frac{1}{2}F'(x) dx \\ &= \left[\frac{1}{2}F(x) \right]_a^b \\ &= \frac{1}{2}[F(b) - F(a)] \\ &= \frac{1}{2}[f(b)^2 - f(a)^2]. \end{aligned}$$

13. Consider $\int_0^1 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_0^1 = \frac{2}{3}$. The corresponding

Riemann Sum using right-hand endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}} \right] \\ &= \frac{1}{n^{3/2}} [\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}]. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}$.

14. Consider $\int_0^1 x^5 dx = \left[\frac{x^6}{6} \right]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\left(\frac{1}{n} \right)^5 + \left(\frac{2}{n} \right)^5 + \cdots + \left(\frac{n}{n} \right)^5 \right] \\ &= \frac{1}{n^6} [1^5 + 2^5 + \cdots + n^5]. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}$.

15. By Theorem 4.8, $0 < f(x) \leq M \Rightarrow \int_a^b f(x) dx \leq \int_a^b M dx = M(b-a)$.

Similarly, $m \leq f(x) \Rightarrow m(b-a) = \int_a^b m dx \leq \int_a^b f(x) dx$.

Thus, $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$. On the interval $[0, 1]$, $1 \leq \sqrt{1+x^4} \leq \sqrt{2}$ and $b-a = 1$.

Thus, $1 \leq \int_0^1 \sqrt{1+x^4} dx \leq \sqrt{2}$. (Note: $\int_0^1 \sqrt{1+x^4} dx \approx 1.0894$)

16. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b-x)} dx$.

Let $u = b - x$, $du = -dx$.

$$\begin{aligned} A &= \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)} (-du) \\ &= \int_0^b \frac{f(b-u)}{f(b-u) + f(u)} du \\ &= \int_0^b \frac{f(b-x)}{f(b-x) + f(x)} dx \end{aligned}$$

Then,

$$\begin{aligned} 2A &= \int_0^b \frac{f(x)}{f(x) + f(b-x)} dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)} dx \\ &= \int_0^b 1 dx = b. \end{aligned}$$

Thus, $A = \frac{b}{2}$.

(b) $b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x} dx = \frac{1}{2}$

(c) $b = 3, f(x) = \sqrt{x}$

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}} dx = \frac{3}{2}$$

17. (a) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1+i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i+1)^3 - i^3$

$$\begin{aligned} \sum_{i=1}^n (3i^2 + 3i + 1) &= \sum_{i=1}^n [(i+1)^3 - i^3] \\ &= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [(n+1)^3 - n^3] \\ &= (n+1)^3 - 1 \end{aligned}$$

Hence, $(n+1)^3 = \sum_{i=1}^n (3i^2 + 3i + 1) + 1$.

(c) $(n+1)^3 - 1 = \sum_{i=1}^n (3i^2 + 3i + 1) = \sum_{i=1}^n 3i^2 + \frac{3(n)(n+1)}{2} + n$

$$\begin{aligned} \Rightarrow \sum_{i=1}^n 3i^2 &= n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n \\ &= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2} \\ &= \frac{2n^3 + 3n^2 + n}{2} \\ &= \frac{n(n+1)(2n+1)}{2} \\ \Rightarrow \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

18. Since $-|f(x)| \leq f(x) \leq |f(x)|$,

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \Rightarrow \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

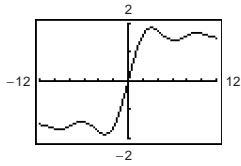
19. (a) $R < I < T < L$

(b) $S(4) = \frac{4-0}{3(4)} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3} \left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4} \right] \approx 5.417$$

$$20. \text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$$

(a)



$$(b) \text{Si}'_i(x) = \frac{\sin x}{x} \quad \text{Si}'(x) = 0 \text{ for } x = 2n\pi$$

 For positive x , $x = (2n - 1)\pi$

 For negative x , $x = 2n\pi$

 Maxima at $\pi, 3\pi, 5\pi, \dots$ and $-2\pi, -4\pi, -6\pi, \dots$

$$(c) \text{Si}''(x) = \frac{x \cos x - \sin x}{x^2} = 0$$

$$x \cos x = \sin x \text{ for } x \approx 4.4934$$

$$\text{Si}(4.4934) \approx 1.6556$$

 (d) Horizontal asymptotes at $y = \pm \frac{\pi}{2}$

$$\lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \text{Si}(x) = -\frac{\pi}{2}$$

94. $\int_0^2 1.75 \sin \frac{\pi t}{2} dt = -\frac{2}{\pi} \left[1.75 \cos \frac{\pi t}{2} \right] = -\frac{2}{\pi} (1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282$ liters

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

95. Trapezoidal Rule ($n = 4$): $\int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{8} \left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.257$

Simpson's Rule ($n = 4$): $\int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{12} \left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.254$

Graphing utility: 0.254

96. Trapezoidal Rule ($n = 4$): $\int_0^1 \frac{x^{3/2}}{3-x^2} dx \approx \frac{1}{8} \left[0 + \frac{2(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{2(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2} \right] \approx 0.172$

Simpson's Rule ($n = 4$): $\int_0^1 \frac{x^{3/2}}{3-x^2} dx \approx \frac{1}{12} \left[0 + \frac{4(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{4(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2} \right] \approx 0.166$

Graphing utility: 0.166

97. Trapezoidal Rule ($n = 4$): $\int_0^{\pi/2} \sqrt{x} \cos x dx \approx 0.637$

Simpson's Rule ($n = 4$): 0.685

Graphing Utility: 0.704

98. Trapezoidal Rule ($n = 4$): $\int_0^{\pi} \sqrt{1+\sin^2 x} dx \approx 3.820$

Simpson's Rule ($n = 4$): 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t} dt = 0$

(c) $L(x) = 1 = \int_1^x \frac{1}{t} dt$ for $x \approx 2.718$

$$\int_1^{2.718} \frac{1}{t} dt = 0.999896$$

(Note: The exact value of x is e , the base of the natural logarithm function.)

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$$L'(1) = 1$$

(d) We first show that $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{u} du$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1} dt$.

$$\text{Then } \int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{ux_1} (x_1 du) = \int_{1/x_1}^1 \frac{1}{u} du = \int_{1/x_1}^1 \frac{1}{u} du.$$

$$\text{Now, } L(x_1 x_2) = \int_1^{x_1 x_2} \frac{1}{t} dt = \int_{1/x_1}^{x_2} \frac{1}{u} du \text{ (using } u = \frac{t}{x_1} \text{)}$$

$$= \int_{1/x_1}^1 \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du$$

$$= \int_1^{x_1} \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du$$

$$= L(x_1) + L(x_2).$$

2. (a) $F(x) = \int_2^x \sin t^2 dt$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	-0.8048	-0.4945	-0.0265	0.0611	0	-0.0867	-0.3743	-0.0312	-0.0576	-0.2769

(b) $G(x) = \frac{1}{x-2} \int_2^x \sin t^2 dt$

x	1.9	1.95	1.99	2.01	2.1
$G(x)$	-0.6106	-0.6873	-0.7436	-0.7697	-0.8671

$$\lim_{x \rightarrow 2} G(x) \approx -0.75$$

$$\begin{aligned} \text{(c) } F'(2) &= \lim_{x \rightarrow 2} \frac{F(x) - F(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{1}{x - 2} \int_2^x \sin t^2 dt \\ &= \lim_{x \rightarrow 2} G(x) \end{aligned}$$

Since $F'(x) = \sin x^2$, $F'(2) = \sin 4 = \lim_{x \rightarrow 2} G(x)$. (**Note:** $\sin 4 \approx -0.7568$)

3. $y = x^4 - 4x^3 + 4x^2$, $[0, 2]$, $c_i = \frac{2i}{n}$

(a) $\Delta x = \frac{2}{n}$, $f(x) = x^4 - 4x^3 + 4x^2$

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^4 - 4 \left(\frac{2i}{n} \right)^3 + 4 \left(\frac{2i}{n} \right)^2 \right] \frac{2}{n} \end{aligned}$$

$$\begin{aligned} \text{(b) } \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^4 - 4 \left(\frac{2i}{n} \right)^3 + 4 \left(\frac{2i}{n} \right)^2 \right] \frac{2}{n} \\ &= \left[\frac{8(n+1)(6n^3 + 9n^2 + n - 1)}{15n^3} - \frac{8(n+1)^2}{n} + \frac{8(n+1)(2n+1)}{3n} \right] \frac{2}{n} \\ &= \left[\frac{8(n+1)(n^3 - n^2 + n - 1)}{15n^3} \right] \frac{2}{n} \end{aligned}$$

(c) $A = \lim_{n \rightarrow \infty} \left[\frac{8(n+1)(n^3 - n^2 + n - 1)}{15n^3} \right] \frac{2}{n} = \frac{16}{15}$

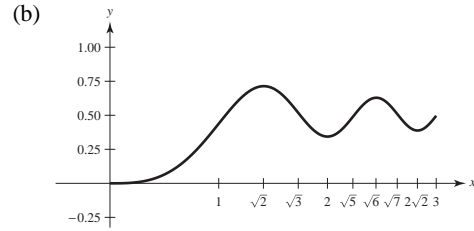
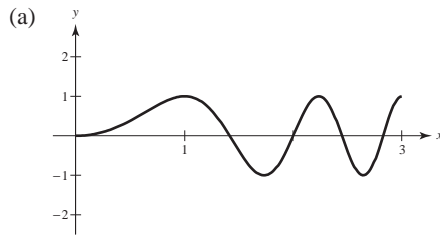
4. $y = \frac{1}{2}x^5 + 2x^3$, $[0, 2]$, $c_i = \frac{2i}{n}$, $\Delta x = \frac{2}{n}$

$$\begin{aligned} \text{(a) } A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{1}{2} \left(\frac{2i}{n} \right)^5 + 2 \left(\frac{2i}{n} \right)^3 \right] \frac{2}{n} \end{aligned}$$

$$\begin{aligned} \text{(b) } \sum_{i=1}^n \left[\frac{1}{2} \left(\frac{2i}{n} \right)^5 + 2 \left(\frac{2i}{n} \right)^3 \right] \frac{2}{n} \\ &= \left[\frac{4(n+1)^2(2n^2 + 2n - 1)}{3n^3} + \frac{4(n+1)^2}{n} \right] \frac{2}{n} \\ &= \left[\frac{8(n+1)^2(5n^2 + 2n - 1)}{3n^4} \right] \end{aligned}$$

(c) $A = \lim_{n \rightarrow \infty} \left[\frac{8(n+1)^2(5n^2 + 2n - 1)}{3n^4} \right] = \frac{40}{3}$

5. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$



The zeros of $y = \sin \frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin \frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}, n \text{ integer.}$

Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n}, n \text{ integer}$

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \text{ and } \sqrt{7}$.

6. (a) $\int_{-1}^1 \cos x \, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2 \cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\int_{-1}^1 \cos x \, dx = \sin x \Big|_{-1}^1 = 2 \sin(1) \approx 1.6829$

Error: $|1.6829 - 1.6758| = 0.0071$

(b) $\int_{-1}^1 \frac{1}{1+x^2} \, dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$

(Note: exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$\int_{-1}^1 p(x) \, dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx \right]_{-1}^1 = \frac{2b}{3} + 2d$

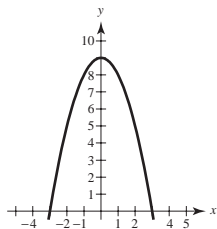
$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$

7. (a) Area = $\int_{-3}^3 (9 - x^2) \, dx = 2 \int_0^3 (9 - x^2) \, dx$

$= 2 \left[9x - \frac{x^3}{3} \right]_0^3$

$= 2[27 - 9] = 36$

(b) Base = 6, height = 9, Area = $\frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$



(c) Let the parabola be given by $y = b^2 - a^2x^2, a, b > 0$.

Area = $2 \int_0^{b/a} (b^2 - a^2x^2) \, dx$

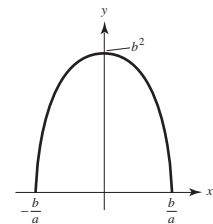
$= 2 \left[b^2x - a^2 \frac{x^3}{3} \right]_0^{b/a}$

$= 2 \left[b^2 \left(\frac{b}{a}\right) - \frac{a^2}{3} \left(\frac{b}{a}\right)^3 \right]$

$= 2 \left[\frac{b^3}{a} - \frac{1}{3} \frac{b^3}{a} \right] = \frac{4}{3} \frac{b^3}{a}$

Base = $\frac{2b}{a}$, height = b^2

Archimedes' Formula: Area = $\frac{2}{3} \left(\frac{2b}{a}\right)(b^2) = \frac{4}{3} \frac{b^3}{a}$



8. Let d be the distance traversed and a be the uniform acceleration. We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$$s(t) = d \text{ when } t = \sqrt{\frac{2d}{a}}.$$

$$\text{The highest speed is } v = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}.$$

The lowest speed is $v = 0$.

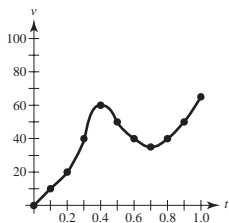
$$\text{The mean speed is } \frac{1}{2}(\sqrt{2ad} + 0) = \sqrt{\frac{ad}{2}}.$$

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

10. (a)



(b) v is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

$$(c) \text{ Average acceleration} = \frac{v(0.4) - v(0)}{0.4 - 0} = \frac{60 - 0}{0.4} = 150 \text{ mi/hr}^2$$

(d) This integral is the total distance traveled in miles.

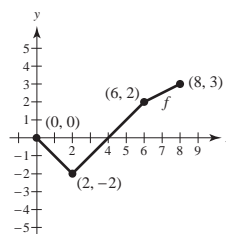
$$\int_0^1 v(t) dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

9. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

$$(c) f(x) = \begin{cases} -x, & 0 \leq x < 2 \\ x - 4, & 2 \leq x < 6 \\ \frac{1}{2}x - 1, & 6 \leq x \leq 8 \end{cases}$$

$$F(x) = \int_0^x f(t) dt = \begin{cases} (-x^2/2), & 0 \leq x < 2 \\ (x^2/2) - 4x + 4, & 2 \leq x < 6 \\ (1/4)x^2 - x - 5, & 6 \leq x \leq 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

$$(d) F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

$$11. \int_0^x f(t)(x-t) dt = \int_0^x xf(t) dt - \int_0^x tf(t) dt = x \int_0^x f(t) dt - \int_0^x tf(t) dt$$

$$\text{Thus, } \frac{d}{dx} \int_0^x f(t)(x-t) dt = xf(x) + \int_0^x f(t) dt - xf(x) = \int_0^x f(t) dt$$

Differentiating the other integral,

$$\frac{d}{dx} \int_0^x \left(\int_0^x f(v) dv \right) dt = \int_0^x f(v) dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x-t) dt = \int_0^x \left(\int_0^x f(v) dv \right) dt + C.$$

Letting $x = 0$, we see that $C = 0$.

12. Consider $F(x) = [f(x)]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. Thus,

$$\begin{aligned} \int_a^b f(x)f'(x) dx &= \int_a^b \frac{1}{2}F'(x) dx \\ &= \left[\frac{1}{2}F(x) \right]_a^b \\ &= \frac{1}{2}[F(b) - F(a)] \\ &= \frac{1}{2}[f(b)^2 - f(a)^2]. \end{aligned}$$

13. Consider $\int_0^1 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_0^1 = \frac{2}{3}$. The corresponding

Riemann Sum using right-hand endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}} \right] \\ &= \frac{1}{n^{3/2}} [\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}]. \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}.$$

$$14. \text{ Consider } \int_0^1 x^5 dx = \left[\frac{x^6}{6} \right]_0^1 = \frac{1}{6}.$$

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\left(\frac{1}{n} \right)^5 + \left(\frac{2}{n} \right)^5 + \cdots + \left(\frac{n}{n} \right)^5 \right] \\ &= \frac{1}{n^6} [1^5 + 2^5 + \cdots + n^5]. \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}.$$

$$15. \text{ By Theorem 4.8, } 0 < f(x) \leq M \Rightarrow \int_a^b f(x) dx \leq \int_a^b M dx = M(b-a).$$

$$\text{Similarly, } m \leq f(x) \Rightarrow m(b-a) = \int_a^b m dx \leq \int_a^b f(x) dx.$$

Thus, $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$. On the interval $[0, 1]$, $1 \leq \sqrt{1+x^4} \leq \sqrt{2}$ and $b-a = 1$.

$$\text{Thus, } 1 \leq \int_0^1 \sqrt{1+x^4} dx \leq \sqrt{2}. \quad (\text{Note: } \int_0^1 \sqrt{1+x^4} dx \approx 1.0894)$$

16. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b-x)} dx$.

Let $u = b - x$, $du = -dx$.

$$\begin{aligned} A &= \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)} (-du) \\ &= \int_0^b \frac{f(b-u)}{f(b-u) + f(u)} du \\ &= \int_0^b \frac{f(b-x)}{f(b-x) + f(x)} dx \end{aligned}$$

Then,

$$\begin{aligned} 2A &= \int_0^b \frac{f(x)}{f(x) + f(b-x)} dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)} dx \\ &= \int_0^b 1 dx = b. \end{aligned}$$

Thus, $A = \frac{b}{2}$.

(b) $b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x} dx = \frac{1}{2}$

(c) $b = 3, f(x) = \sqrt{x}$

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}} dx = \frac{3}{2}$$

17. (a) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1+i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i+1)^3 - i^3$

$$\begin{aligned} \sum_{i=1}^n (3i^2 + 3i + 1) &= \sum_{i=1}^n [(i+1)^3 - i^3] \\ &= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [(n+1)^3 - n^3] \\ &= (n+1)^3 - 1 \end{aligned}$$

Hence, $(n+1)^3 = \sum_{i=1}^n (3i^2 + 3i + 1) + 1$.

(c) $(n+1)^3 - 1 = \sum_{i=1}^n (3i^2 + 3i + 1) = \sum_{i=1}^n 3i^2 + \frac{3(n)(n+1)}{2} + n$

$$\begin{aligned} \Rightarrow \sum_{i=1}^n 3i^2 &= n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n \\ &= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2} \\ &= \frac{2n^3 + 3n^2 + n}{2} \\ &= \frac{n(n+1)(2n+1)}{2} \\ \Rightarrow \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

18. Since $-|f(x)| \leq f(x) \leq |f(x)|$,

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \Rightarrow \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

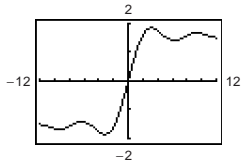
19. (a) $R < I < T < L$

(b) $S(4) = \frac{4-0}{3(4)} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3} \left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4} \right] \approx 5.417$$

20. $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$

(a)



(b) $\text{Si}'_i(x) = \frac{\sin x}{x}$ $\text{Si}'(x) = 0$ for $x = 2n\pi$

For positive x , $x = (2n - 1)\pi$

For negative x , $x = 2n\pi$

Maxima at $\pi, 3\pi, 5\pi, \dots$ and $-2\pi, -4\pi, -6\pi, \dots$

(c) $\text{Si}''(x) = \frac{x \cos x - \sin x}{x^2} = 0$

$x \cos x = \sin x$ for $x \approx 4.4934$

$\text{Si}(4.4934) \approx 1.6556$

(d) Horizontal asymptotes at $y = \pm \frac{\pi}{2}$

$$\lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \text{Si}(x) = -\frac{\pi}{2}$$