

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

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CHAPTER 5

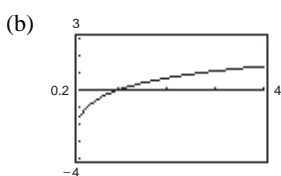
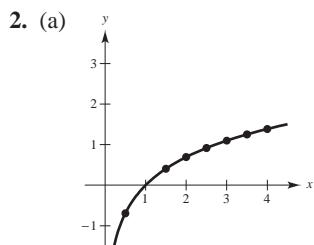
Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t} dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t} dt = -\int_{0.5}^1 \frac{1}{t} dt$



The graphs are identical.

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t} dt \approx 3.8067$

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t} dt \approx 2.1163$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t} dt \approx -0.2231$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t} dt \approx -0.5108$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

8. $f(x) = -\ln x$

Reflection in the x -axis

Matches (d)

9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

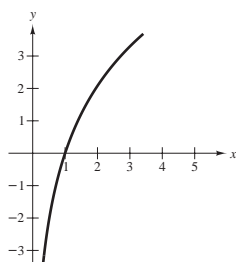
10. $f(x) = -\ln(-x)$

Reflection in the y -axis and the x -axis

Matches (c)

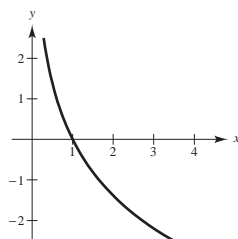
11. $f(x) = 3 \ln x$

Domain: $x > 0$



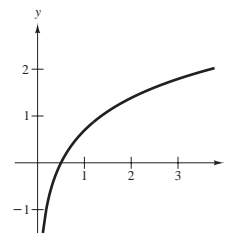
12. $f(x) = -2 \ln x$

Domain: $x > 0$

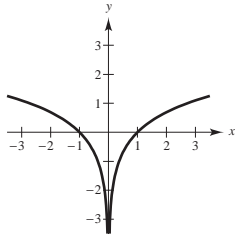


13. $f(x) = \ln 2x$

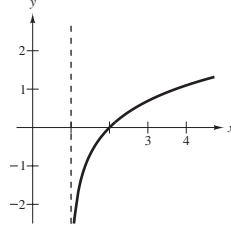
Domain: $x > 0$



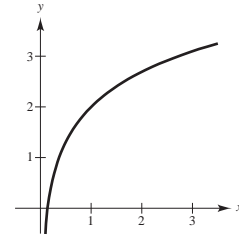
14. $f(x) = \ln|x|$

 Domain: $x \neq 0$


15. $f(x) = \ln(x - 1)$

 Domain: $x > 1$


16. $g(x) = 2 + \ln x$

 Domain: $x > 0$


17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

20. $\ln \sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2} \ln 2$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

22. $\ln(xyz) = \ln x + \ln y + \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

24. $\ln \sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right) \ln(a-1)$

25. $\ln \left(\frac{x^2 - 1}{x^3}\right)^3 = 3[\ln(x^2 - 1) - \ln x^3]$
 $= 3[\ln(x+1) + \ln(x-1) - 3 \ln x]$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

27. $\ln z(z-1)^2 = \ln z + \ln(z-1)^2$
 $= \ln z + 2 \ln(z-1)$

28. $\ln \frac{1}{e} = \ln 1 - \ln e = -1$

29. $\ln(x-2) - \ln(x+2) = \ln \frac{x-2}{x+2}$

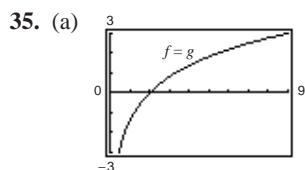
30. $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$
 $= \ln \frac{x^3 y^2}{z^4}$

31. $\frac{1}{3}[2 \ln(x+3) + \ln x - \ln(x^2 - 1)] = \frac{1}{3} \ln \frac{x(x+3)^2}{x^2 - 1}$
 $= \ln \sqrt[3]{\frac{x(x+3)^2}{x^2 - 1}}$

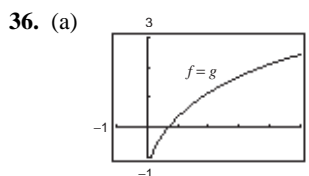
32. $2[\ln x - \ln(x+1) - \ln(x-1)] = 2 \ln \frac{x}{(x+1)(x-1)}$
 $= \ln \left(\frac{x}{x^2 - 1}\right)^2$

33. $2 \ln 3 - \frac{1}{2} \ln(x^2 + 1) = \ln 9 - \ln \sqrt{x^2 + 1} = \ln \frac{9}{\sqrt{x^2 + 1}}$

34. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x+1) - \ln(x-1)] = \frac{3}{2} \ln \frac{x^2 + 1}{(x+1)(x-1)} = \ln \sqrt{\left(\frac{x^2 + 1}{x^2 - 1}\right)^3}$



(b) $f(x) = \ln \frac{x^2}{4} = \ln x^2 - \ln 4 = 2 \ln x - \ln 4 = g(x)$
 since $x > 0$.



(b) $f(x) = \ln \sqrt{x(x^2 + 1)} = \frac{1}{2} \ln[x(x^2 + 1)]$
 $= \frac{1}{2} [\ln x + \ln(x^2 + 1)] = g(x)$

37. $\lim_{x \rightarrow 3^+} \ln(x - 3) = -\infty$

38. $\lim_{x \rightarrow 6^-} \ln(6 - x) = -\infty$

39. $\lim_{x \rightarrow 2^-} \ln[x^2(3 - x)] = \ln 4$
 ≈ 1.3863

40. $\lim_{x \rightarrow 5^+} \ln \frac{x}{\sqrt{x - 4}} = \ln 5 \approx 1.6094$

41. $y = \ln x^3 = 3 \ln x$

42. $y = \ln x^{3/2} = \frac{3}{2} \ln x$

$$y' = \frac{3}{x}$$

$$y' = \frac{3}{2x}$$

Slope at (1, 0) is $\frac{3}{1} = 3$.

Slope at (1, 0) is $\frac{3}{2}$.

$$y - 0 = 3(x - 1)$$

$$y - 0 = \frac{3}{2}(x - 1)$$

$$y = 3x - 3 \quad \text{Tangent line}$$

$$y = \frac{3}{2}x - \frac{3}{2} \quad \text{Tangent line}$$

43. $y = \ln x^2 = 2 \ln x$

$$y' = \frac{2}{x}$$

Slope at (1, 0) is 2.

$$y - 0 = 2(x - 1)$$

$$y = 2x - 2 \quad \text{Tangent line}$$

44. $y = \ln x^{1/2} = \frac{1}{2} \ln x$

$$y' = \frac{1}{2x}$$

At (1, 0), slope is $\frac{1}{2}$.

$$y - 0 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x - \frac{1}{2} \quad \text{Tangent line}$$

45. $g(x) = \ln x^2 = 2 \ln x$

$$g'(x) = \frac{2}{x}$$

46. $h(x) = \ln(2x^2 + 1)$

$$h'(x) = \frac{1}{2x^2 + 1}(4x) = \frac{4x}{2x^2 + 1}$$

47. $y = (\ln x)^4$

$$\frac{dy}{dx} = 4(\ln x)^3 \left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

48. $y = x \ln x$

$$\frac{dy}{dx} = x \left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

49. $y = \ln[x\sqrt{x^2 - 1}] = \ln x + \frac{1}{2} \ln(x^2 - 1)$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x(x^2 - 1)}$$

50. $y = \ln \sqrt{x^2 - 4} = \frac{1}{2} \ln(x^2 - 4)$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{2x}{x^2 - 4}\right) = \frac{x}{x^2 - 4}$$

51. $f(x) = \ln \frac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$$

52. $f(x) = \ln\left(\frac{2x}{x + 3}\right) = \ln 2x - \ln(x + 3)$

$$f'(x) = \frac{1}{x} - \frac{1}{x + 3} = \frac{3}{x(x + 3)}$$

53. $g(t) = \frac{\ln t}{t^2}$

$$g'(t) = \frac{t^2(1/t) - 2t \ln t}{t^4} = \frac{1 - 2 \ln t}{t^3}$$

54. $h(t) = \frac{\ln t}{t}$

$$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$$

55. $y = \ln(\ln x^2)$

$$\frac{dy}{dx} = \frac{1}{\ln x^2} \frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

56. $y = \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

57. $y = \ln \sqrt{\frac{x+1}{x-1}} = \frac{1}{2}[\ln(x+1) - \ln(x-1)]$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x+1} - \frac{1}{x-1} \right] = \frac{1}{1-x^2}$$

58. $y = \ln \sqrt[3]{\frac{x-1}{x+1}} = \frac{1}{3}[\ln(x-1) - \ln(x+1)]$

$$y' = \frac{1}{3} \left[\frac{1}{x-1} - \frac{1}{x+1} \right] = \frac{1}{3} \frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$$

59. $f(x) = \ln \frac{\sqrt{4+x^2}}{x} = \frac{1}{2} \ln(4+x^2) - \ln x$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$$

60. $f(x) = \ln(x + \sqrt{4+x^2})$

$$f'(x) = \frac{1}{x + \sqrt{4+x^2}} \left(1 + \frac{x}{\sqrt{4+x^2}} \right) = \frac{1}{\sqrt{4+x^2}}$$

61. $y = \frac{-\sqrt{x^2+1}}{x} + \ln(x + \sqrt{x^2+1})$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-x(x/\sqrt{x^2+1}) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x + \sqrt{x^2+1}} \right) \left(1 + \frac{x}{\sqrt{x^2+1}} \right) \\ &= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x + \sqrt{x^2+1}} \right) \left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \right) \\ &= \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2} \end{aligned}$$

62. $y = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln \left(\frac{2 + \sqrt{x^2+4}}{x} \right) = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln(2 + \sqrt{x^2+4}) + \frac{1}{4} \ln x$

$$\frac{dy}{dx} = \frac{-2x^2(x/\sqrt{x^2+4}) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4} \left(\frac{1}{2 + \sqrt{x^2+4}} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x}$$

Note that $\frac{1}{2 + \sqrt{x^2+4}} = \frac{1}{2 + \sqrt{x^2+4}} \cdot \frac{2 - \sqrt{x^2+4}}{2 - \sqrt{x^2+4}} = \frac{2 - \sqrt{x^2+4}}{-x^2}$.

Hence,
$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4} \frac{(2 - \sqrt{x^2+4})}{-x^2} \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x} \\ &= \frac{-1 + (1/2)(2 - \sqrt{x^2+4})}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} \\ &= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3}. \end{aligned}$$

63. $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

64. $y = \ln|\csc x|$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

65. $y = \ln \left| \frac{\cos x}{\cos x - 1} \right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1}$$

$$= -\tan x + \frac{\sin x}{\cos x - 1}$$

$$66. y = \ln|\sec x + \tan x|$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x \end{aligned}$$

$$67. y = \ln \left| \frac{-1 + \sin x}{2 + \sin x} \right|$$

$$\begin{aligned} &= \ln|-1 + \sin x| - \ln|2 + \sin x| \\ \frac{dy}{dx} &= \frac{\cos x}{-1 + \sin x} - \frac{\cos x}{2 + \sin x} \\ &= \frac{3 \cos x}{(\sin x - 1)(\sin x + 2)} \end{aligned}$$

$$68. y = \ln \sqrt{2 + \cos^2 x}$$

$$\begin{aligned} &= \frac{1}{2} \ln(2 + \cos^2 x) \\ y' &= \frac{1}{2} \frac{-2 \cos x \sin x}{2 + \cos^2 x} \\ &= \frac{-\cos x \sin x}{2 + \cos^2 x} \end{aligned}$$

$$69. f(x) = \int_2^{\ln(2x)} (t+1) dt$$

$$f'(x) = [\ln(2x) + 1] \left(\frac{1}{x} \right) = \frac{\ln(2x) + 1}{x}$$

Second solution:

$$\begin{aligned} f(x) &= \int_2^{\ln(2x)} (t+1) dt \\ &= \left[\frac{t^2}{2} + t \right]_2^{\ln(2x)} = \left[\frac{[\ln(2x)]^2}{2} + \ln(2x) \right] - [2 + 2] \\ f'(x) &= \frac{1}{2} 2 \ln(2x) \frac{1}{x} + \frac{1}{2x}(2) = \frac{\ln(2x) + 1}{x} \end{aligned}$$

$$70. g(x) = \int_1^{\ln x} (t^2 + 3) dt$$

$$g'(x) = [(\ln x)^2 + 3] \frac{d}{dx}(\ln x) = \frac{(\ln x)^2 + 3}{x}$$

(Second Fundamental Theorem of Calculus)

$$71. (a) y = 3x^2 - \ln x, \quad (1, 3)$$

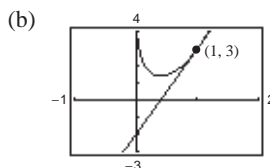
$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

$$\text{When } x = 1, \frac{dy}{dx} = 5.$$

$$\text{Tangent line: } y - 3 = 5(x - 1)$$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$



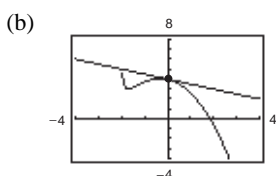
$$72. (a) y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$$

$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x + 1} \left(\frac{1}{2} \right) = -2x - \frac{1}{x + 2}$$

$$\text{When } x = 0, \frac{dy}{dx} = -\frac{1}{2}.$$

$$\text{Tangent line: } y - 4 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 4$$



$$73. (a) f(x) = \ln \sqrt{1 + \sin^2 x}$$

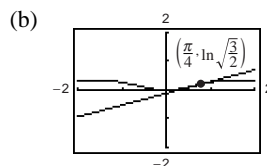
$$= \frac{1}{2} \ln(1 + \sin^2 x), \quad \left(\frac{\pi}{4}, \ln \sqrt{\frac{3}{2}} \right)$$

$$f'(x) = \frac{2 \sin x \cos x}{2(1 + \sin^2 x)} = \frac{\sin x \cos x}{1 + \sin^2 x}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{(\sqrt{2}/2)(\sqrt{2}/2)}{(3/2)} = \frac{1}{3}$$

$$\text{Tangent line: } y - \ln \sqrt{\frac{3}{2}} = \frac{1}{3} \left(x - \frac{\pi}{4} \right)$$

$$y = \frac{1}{3}x + \frac{1}{2} \ln\left(\frac{3}{2}\right) - \frac{\pi}{12}$$



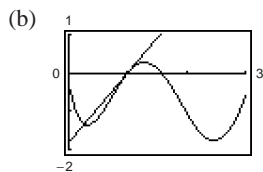
74. (a) $f(x) = \sin(2x) \ln(x^2) = 2 \sin(2x) \ln x, \quad (1, 0)$

$$f'(x) = 4 \cos(2x) \ln x + \frac{2 \sin(2x)}{x}$$

$$f'(1) = 2 \sin(2)$$

$$\text{Tangent line: } y - 0 = 2 \sin(2)(x - 1)$$

$$y = 2 \sin(2)x - 2 \sin(2)$$



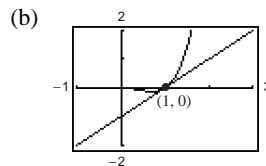
75. (a) $f(x) = x^3 \ln x, \quad (1, 0)$

$$f'(x) = 3x^2 \ln x + x^2$$

$$f'(1) = 1$$

$$\text{Tangent line: } y - 0 = 1(x - 1)$$

$$y = x - 1$$



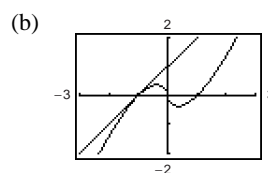
76. (a) $f(x) = \frac{1}{2}x \ln(x^2), \quad (-1, 0)$

$$f'(x) = \frac{1}{2} \ln(x^2) + \frac{1}{2}x \left(\frac{2x}{x^2} \right) = \frac{1}{2} \ln(x^2) + 1$$

$$f'(-1) = 1$$

$$\text{Tangent line: } y - 0 = 1(x + 1)$$

$$y = x + 1$$



77. $x^2 - 3 \ln y + y^2 = 10$

$$2x - \frac{3}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx} \left(\frac{3}{y} - 2y \right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

79. $x + y - 1 = \ln(x^2 + y^2), \quad (1, 0)$

$$1 + y' = \frac{2x + 2yy'}{x^2 + y^2}$$

$$x^2 + y^2 + (x^2 + y^2)y' = 2x + 2yy'$$

$$\text{At } (1, 0): 1 + y' = 2$$

$$y' = 1$$

$$\text{Tangent line: } y = x - 1$$

78. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y + 5xy}{x} \right)$$

80. $y^2 + \ln(xy) = 2, \quad (e, 1)$

$$2yy' + \frac{xy' + y}{xy} = 0$$

$$2xy^2y' + xy' + y = 0$$

$$\text{At } (e, 1): 2ey' + ey' + 1 = 0$$

$$y' = \frac{-1}{3e}$$

$$\text{Tangent line: } y - 1 = \frac{-1}{3e}(x - e)$$

$$y = \frac{-1}{3e}x + \frac{4}{3}$$

81. $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

83. $y = \frac{x^2}{2} - \ln x$

Domain: $x > 0$

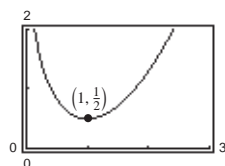
$$y' = x - \frac{1}{x}$$

$$= \frac{(x+1)(x-1)}{x}$$

$$= 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

Relative minimum: $\left(1, \frac{1}{2}\right)$



82. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x \ln x - 4x - x(-3 + \ln x) = 0$$

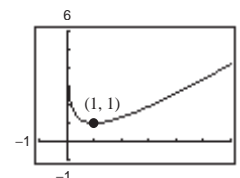
84. $y = x - \ln x$

Domain: $x > 0$

$$y' = 1 - \frac{1}{x} = 0 \text{ when } x = 1.$$

$$y'' = \frac{1}{x^2} > 0$$

Relative minimum: $(1, 1)$



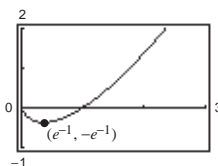
85. $y = x \ln x$

Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

Relative minimum: $(e^{-1}, -e^{-1})$



86. $y = \frac{\ln x}{x}$

Domain: $x > 0$

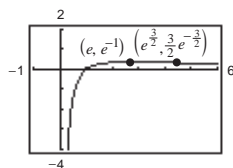
$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4}$$

$$= \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

Relative maximum: (e, e^{-1})

Point of inflection: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$



87. $y = \frac{x}{\ln x}$

Domain: $0 < x < 1, x > 1$

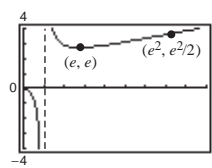
$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x) \ln x}{(\ln x)^4}$$

$$= \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $\left(e^2, \frac{e^2}{2}\right)$



88. $y = x^2 \ln \frac{x}{4}$, Domain: $x > 0$

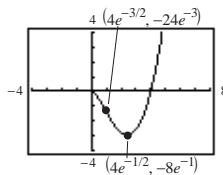
$$y' = x^2 \left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x \left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when:}$$

$$-1 = 2 \ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x \left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$$y'' = 0 \text{ when } x = 4e^{-3/2}$$

 Relative minimum: $(4e^{-1/2}, -8e^{-1})$

 Point of inflection: $(4e^{-3/2}, -24e^{-3})$


89. $f(x) = \ln x$, $f(1) = 0$

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

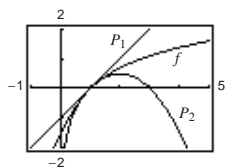
$$= (x - 1) - \frac{1}{2}(x - 1)^2, \quad P_2(1) = 0$$

$$P_1'(x) = 1, \quad P_1'(1) = 1$$

$$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$$

$$P_2''(x) = -1, \quad P_2''(1) = -1$$

The values of f , P_1 , P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



90. $f(x) = x \ln x$, $f(1) = 0$

$$f'(x) = 1 + \ln x, \quad f'(1) = 1$$

$$f''(x) = \frac{1}{x}, \quad f''(1) = 1$$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

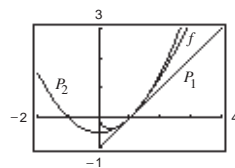
$$= (x - 1) + \frac{1}{2}(x - 1)^2, \quad P_2(1) = 0$$

$$P_1'(x) = 1, \quad P_1'(1) = 1$$

$$P_2'(x) = 1 + (x - 1) = x, \quad P_2'(1) = 1$$

$$P_2''(x) = x, \quad P_2''(1) = 1$$

The values of f , P_1 , P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



91. Find x such that $\ln x = -x$.

$$f(x) = \ln x + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{1 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	-0.1931	-0.0076	-0.0001

 Approximate root: $x = 0.567$

92. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{4 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	-0.3069	-0.0049	0.0000

 Approximate root: $x = 2.208$

$$93. \quad y = x\sqrt{x^2 - 1}$$

$$\ln y = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$\frac{dy}{dx} = y \left[\frac{2x^2 - 1}{x(x^2 - 1)} \right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$95. \quad y = \frac{x^2 \sqrt{3x - 2}}{(x - 1)^2}$$

$$\ln y = 2 \ln x + \frac{1}{2} \ln(3x - 2) - 2 \ln(x - 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} + \frac{3}{2(3x - 2)} - \frac{2}{x - 1}$$

$$\begin{aligned} \frac{dy}{dx} &= y \left[\frac{3x^2 - 15x + 8}{2x(3x - 2)(x - 1)} \right] \\ &= \frac{3x^3 - 15x^2 + 8x}{2(x - 1)^3 \sqrt{3x - 2}} \end{aligned}$$

$$97. \quad y = \frac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$$

$$\ln y = \ln x + \frac{3}{2} \ln(x - 1) - \frac{1}{2} \ln(x + 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{3}{2} \left(\frac{1}{x - 1} \right) - \frac{1}{2} \left(\frac{1}{x + 1} \right)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{2} \left[\frac{2}{x} + \frac{3}{x - 1} - \frac{1}{x + 1} \right] \\ &= \frac{y}{2} \left[\frac{4x^2 + 4x - 2}{x(x^2 - 1)} \right] = \frac{(2x^2 + 2x - 1)\sqrt{x - 1}}{(x + 1)^{3/2}} \end{aligned}$$

99. Answers will vary. See Theorems 5.1 and 5.2.

$$101. \quad g(x) = \ln f(x), \quad f(x) > 0$$

$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

$$94. \quad y = \sqrt{(x - 1)(x - 2)(x - 3)}$$

$$\ln y = \frac{1}{2} [\ln(x - 1) + \ln(x - 2) + \ln(x - 3)]$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{2} \left[\frac{1}{x - 1} + \frac{1}{x - 2} + \frac{1}{x - 3} \right]$$

$$= \frac{1}{2} \left[\frac{3x^2 - 12x + 11}{(x - 1)(x - 2)(x - 3)} \right]$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2 - 12x + 11}{2y} \\ &= \frac{3x^2 - 12x + 11}{2\sqrt{(x - 1)(x - 2)(x - 3)}} \end{aligned}$$

$$96. \quad y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\ln y = \frac{1}{2} [\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{2} \left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \right]$$

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{\frac{x^2 - 1}{x^2 + 1}} \left[\frac{2x}{x^4 - 1} \right] \\ &= \frac{(x^2 - 1)^{1/2} 2x}{(x^2 + 1)^{1/2} (x^2 - 1)(x^2 + 1)} \\ &= \frac{2x}{(x^2 + 1)^{3/2} (x^2 - 1)^{1/2}} \end{aligned}$$

$$98. \quad y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$$

$$\ln y = \ln(x + 1) + \ln(x + 2) - \ln(x - 1) - \ln(x - 2)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x + 1} + \frac{1}{x + 2} - \frac{1}{x - 1} - \frac{1}{x - 2}$$

$$\begin{aligned} \frac{dy}{dx} &= y \left[\frac{-2}{x^2 - 1} + \frac{-4}{x^2 - 4} \right] = y \left[\frac{-6x^2 + 12}{(x^2 - 1)(x^2 - 4)} \right] \\ &= \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)} \cdot \frac{-6(x^2 - 2)}{(x + 1)(x - 1)(x + 2)(x - 2)} \\ &= -\frac{6(x^2 - 2)}{(x - 1)^2(x - 2)^2} \end{aligned}$$

100. The base of the natural logarithmic function is e .

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up).
 $g(x) = \ln(x^2 + 1)$ is not concave up.

102. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \frac{2}{x} = 0$ for $x = 2$.

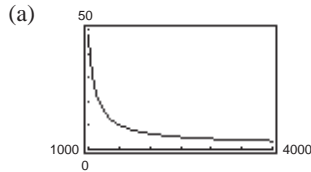
103. False

$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

104. False; π is a constant.

$$\frac{d}{dx}[\ln \pi] = 0$$

105. $t = \frac{5.315}{-6.7968 + \ln x}, \quad 1000 < x$



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280,178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384,642.00$$

(d) $\frac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2} \left(\frac{1}{x}\right)$

$$= -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

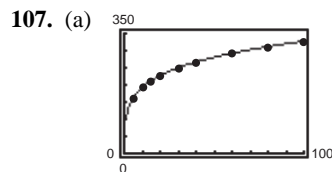
When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

(e) There are two obvious benefits to paying a higher monthly payment:

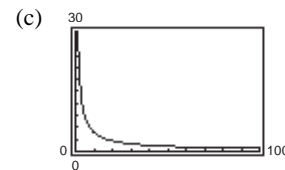
1. The term is lower
2. The total amount paid is lower.

106. $\beta = 10 \log_{10} \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} \ln \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} [\ln I + 16 \ln 10] = 160 + 10 \log_{10} I$

$$\beta(10^{-10}) = \frac{10}{\ln 10} [\ln 10^{-10} + 16 \ln 10] = \frac{10}{\ln 10} [-10 \ln 10 + 16 \ln 10] = \frac{10}{\ln 10} [6 \ln 10] = 60 \text{ decibels}$$



(b) $T'(p) = \frac{34.96}{p} + \frac{3.955}{\sqrt{p}}$
 $T'(10) \approx 4.75 \text{ deg/lb/in}^2$
 $T'(70) \approx 0.97 \text{ deg/lb/in}^2$



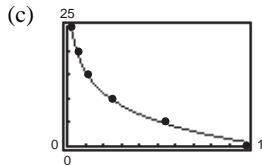
$$\lim_{p \rightarrow \infty} T'(p) = 0$$

108. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474 \ln p.$$

[Note: Fit a line to the data $(x, y) = (\ln p, h)$.]



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmospheres.

(f) $h = 0.8627 - 6.4474 \ln p$

$$1 = -6.4474 \frac{1}{p} \frac{dp}{dh} \quad (\text{implicit differentiation})$$

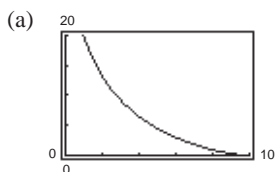
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.5264$ and $dp/dh = -0.0816$ atmos/km.

For $h = 20$, $p = 0.0514$ and $dp/dh = -0.0080$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

$$109. y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10 [\ln(10 + \sqrt{100 - x^2}) - \ln x] - \sqrt{100 - x^2}$$

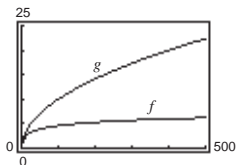


(c) $\lim_{x \rightarrow 10^-} \frac{dy}{dx} = 0$

$$\begin{aligned} (b) \frac{dy}{dx} &= 10 \left[\frac{-x}{\sqrt{100 - x^2}(10 + \sqrt{100 - x^2})} - \frac{1}{x} \right] + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1 \right] - \frac{10}{x} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} \\ &= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x} \\ &= \frac{x(10 - \sqrt{100 - x^2})}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x} \end{aligned}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

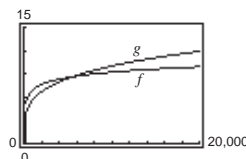
110. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x}, g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x .

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



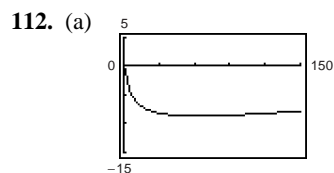
$$f'(x) = \frac{1}{x}, g'(x) = \frac{1}{4\sqrt[3]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x . $f(x) = \ln x$ increases very slowly for “large” values of x .

111. $y = \ln x$

$$y' = \frac{1}{x} > 0 \text{ for } x > 0.$$

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.



(b) Using a graphing utility, there is a relative minimum at $(64, -8.6355)$.

(c) $f'(x) = \frac{1}{2\sqrt{x}} - \frac{4}{x} = 0$

$$2\sqrt{x} = \frac{x}{4}$$

$$8\sqrt{x} = x$$

$$64x = x^2$$

$$x = 64$$

By the First Derivative Test, $x = 64$ is a relative minimum.

Section 5.2 The Natural Logarithmic Function: Integration

1. $\int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln|x| + C$

2. $\int \frac{10}{x} dx = 10 \int \frac{1}{x} dx = 10 \ln|x| + C$

3. $u = x + 1, du = dx$

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

4. $u = x - 5, du = dx$

$$\int \frac{1}{x-5} dx = \ln|x-5| + C$$

5. $u = 3 - 2x, du = -2 dx$

$$\begin{aligned} \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{1}{3-2x} (-2) dx \\ &= -\frac{1}{2} \ln|3-2x| + C \end{aligned}$$

6. $\int \frac{1}{3x+2} dx = \frac{1}{3} \int \frac{1}{3x+2} (3) dx$
$$= \frac{1}{3} \ln|3x+2| + C$$

7. $u = x^2 + 1, du = 2x dx$

$$\begin{aligned} \int \frac{x}{x^2+1} dx &= \frac{1}{2} \int \frac{1}{x^2+1} (2x) dx \\ &= \frac{1}{2} \ln(x^2+1) + C \\ &= \ln\sqrt{x^2+1} + C \end{aligned}$$

8. $u = 3 - x^3, du = -3x^2 dx$

$$\begin{aligned} \int \frac{x^2}{3-x^3} dx &= -\frac{1}{3} \int \frac{1}{3-x^3} (-3x^2) dx \\ &= -\frac{1}{3} \ln|3-x^3| + C \end{aligned}$$

9. $\int \frac{x^2-4}{x} dx = \int \left(x - \frac{4}{x}\right) dx$
$$= \frac{x^2}{2} - 4 \ln|x| + C$$

10. $u = 9 - x^2, du = -2x dx$

$$\begin{aligned} \int \frac{x}{\sqrt{9-x^2}} dx &= -\frac{1}{2} \int (9-x^2)^{-1/2} (-2x) dx \\ &= -\sqrt{9-x^2} + C \end{aligned}$$

11. $u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3) dx$

$$\begin{aligned} \int \frac{x^2+2x+3}{x^3+3x^2+9x} dx &= \frac{1}{3} \int \frac{3(x^2+2x+3)}{x^3+3x^2+9x} dx \\ &= \frac{1}{3} \ln|x^3+3x^2+9x| + C \end{aligned}$$

12. $\int \frac{x(x+2)}{x^3+3x^2-4} dx = \frac{1}{3} \int \frac{3x^2+6x}{x^3+3x^2-4} dx$ ($u = x^3 + 3x^2 - 4$)
$$= \frac{1}{3} \ln|x^3+3x^2-4| + C$$

13. $\int \frac{x^2-3x+2}{x+1} dx = \int \left(x-4 + \frac{6}{x+1}\right) dx$
$$= \frac{x^2}{2} - 4x + 6 \ln|x+1| + C$$

14. $\int \frac{2x^2+7x-3}{x-2} dx = \int \left(2x+11 + \frac{19}{x-2}\right) dx$
$$= x^2 + 11x + 19 \ln|x-2| + C$$

15. $\int \frac{x^3-3x^2+5}{x-3} dx = \int \left(x^2 + \frac{5}{x-3}\right) dx$
$$= \frac{x^3}{3} + 5 \ln|x-3| + C$$

16. $\int \frac{x^3-6x-20}{x+5} dx = \int \left(x^2-5x+19 - \frac{115}{x+5}\right) dx$
$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115 \ln|x+5| + C$$

17. $\int \frac{x^4+x-4}{x^2+2} dx = \int \left(x^2-2 + \frac{x}{x^2+2}\right) dx$
$$= \frac{x^3}{3} - 2x + \frac{1}{2} \ln(x^2+2) + C$$

18. $\int \frac{x^3-3x^2+4x-9}{x^2+3} dx = \int \left(-3+x + \frac{x}{x^2+3}\right) dx$
$$= -3x + \frac{x^2}{2} + \frac{1}{2} \ln(x^2+3) + C$$

$$19. u = \ln x, du = \frac{1}{x} dx$$

$$\int \frac{(\ln x)^2}{x} dx = \frac{1}{3}(\ln x)^3 + C$$

$$21. u = x + 1, du = dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{x+1}} dx &= \int (x+1)^{-1/2} dx \\ &= 2(x+1)^{1/2} + C \\ &= 2\sqrt{x+1} + C \end{aligned}$$

$$\begin{aligned} 23. \int \frac{2x}{(x-1)^2} dx &= \int \frac{2x-2+2}{(x-1)^2} dx \\ &= \int \frac{2(x-1)}{(x-1)^2} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \ln|x-1| - \frac{2}{x-1} + C \end{aligned}$$

$$25. u = 1 + \sqrt{2x}, du = \frac{1}{\sqrt{2x}} dx \Rightarrow (u-1) du = dx$$

$$\begin{aligned} \int \frac{1}{1+\sqrt{2x}} dx &= \int \frac{(u-1)}{u} du = \int \left(1 - \frac{1}{u}\right) du \\ &= u - \ln|u| + C_1 \\ &= (1 + \sqrt{2x}) - \ln|1 + \sqrt{2x}| + C_1 \\ &= \sqrt{2x} - \ln(1 + \sqrt{2x}) + C \end{aligned}$$

where $C = C_1 + 1$.

$$27. u = \sqrt{x} - 3, du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2(u+3) du = dx$$

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{x}-3} dx &= 2 \int \frac{(u+3)^2}{u} du \\ &= 2 \int \frac{u^2 + 6u + 9}{u} du = 2 \int \left(u + 6 + \frac{9}{u}\right) du \\ &= 2 \left[\frac{u^2}{2} + 6u + 9 \ln|u| \right] + C_1 \\ &= u^2 + 12u + 18 \ln|u| + C_1 \\ &= (\sqrt{x}-3)^2 + 12(\sqrt{x}-3) + 18 \ln|\sqrt{x}-3| + C_1 \\ &= x + 6\sqrt{x} + 18 \ln|\sqrt{x}-3| + C \end{aligned}$$

where $C = C_1 - 27$.

$$\begin{aligned} 20. \int \frac{1}{x \ln(x^3)} dx &= \frac{1}{3} \int \frac{1}{\ln x} \cdot \frac{1}{x} dx \\ &= \frac{1}{3} \ln|\ln|x|| + C \end{aligned}$$

$$22. u = 1 + x^{1/3}, du = \frac{1}{3x^{2/3}} dx$$

$$\begin{aligned} \int \frac{1}{x^{2/3}(1+x^{1/3})} dx &= 3 \int \frac{1}{1+x^{1/3}} \left(\frac{1}{3x^{2/3}}\right) dx \\ &= 3 \ln|1+x^{1/3}| + C \end{aligned}$$

$$\begin{aligned} 24. \int \frac{x(x-2)}{(x-1)^3} dx &= \int \frac{x^2-2x+1-1}{(x-1)^3} dx \\ &= \int \frac{(x-1)^2}{(x-1)^3} dx - \int \frac{1}{(x-1)^3} dx \\ &= \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^3} dx \\ &= \ln|x-1| + \frac{1}{2(x-1)^2} + C \end{aligned}$$

$$26. u = 1 + \sqrt{3x}, du = \frac{3}{2\sqrt{3x}} dx \Rightarrow dx = \frac{2}{3}(u-1) du$$

$$\begin{aligned} \int \frac{1}{1+\sqrt{3x}} dx &= \int \frac{1}{u} \frac{2}{3}(u-1) du \\ &= \frac{2}{3} \int \left(1 - \frac{1}{u}\right) du \\ &= \frac{2}{3} [u - \ln|u|] + C \\ &= \frac{2}{3} [1 + \sqrt{3x} - \ln(1 + \sqrt{3x})] + C \\ &= \frac{2}{3} \sqrt{3x} - \frac{2}{3} \ln(1 + \sqrt{3x}) + C_1 \end{aligned}$$

$$28. u = x^{1/3} - 1, du = \frac{1}{3x^{2/3}} dx \Rightarrow dx = 3(u + 1)^2 du$$

$$\begin{aligned} \int \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1} dx &= \int \frac{u + 1}{u} 3(u + 1)^2 du \\ &= 3 \int \frac{u + 1}{u} (u^2 + 2u + 1) du \\ &= 3 \int \left(u^2 + 3u + 3 + \frac{1}{u} \right) du \\ &= 3 \left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u| \right] + C \\ &= 3 \left[\frac{(x^{1/3} - 1)^3}{3} + \frac{3(x^{1/3} - 1)^2}{2} + 3(x^{1/3} - 1) + \ln|x^{1/3} - 1| \right] + C \\ &= 3 \ln|x^{1/3} - 1| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1 \end{aligned}$$

$$29. \int \frac{\cos \theta}{\sin \theta} d\theta = \ln|\sin \theta| + C$$

$(u = \sin \theta, du = \cos \theta d\theta)$

$$30. \int \tan 5\theta d\theta = \frac{1}{5} \int \frac{5 \sin 5\theta}{\cos 5\theta} d\theta$$

$$= -\frac{1}{5} \ln|\cos 5\theta| + C$$

$$31. \int \csc 2x dx = \frac{1}{2} \int (\csc 2x)(2) dx$$

$$= -\frac{1}{2} \ln|\csc 2x + \cot 2x| + C$$

$$32. \int \sec \frac{x}{2} dx = 2 \int \sec \frac{x}{2} \left(\frac{1}{2} \right) dx$$

$$= 2 \ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + C$$

$$33. \int \frac{\cos t}{1 + \sin t} dt = \ln|1 + \sin t| + C$$

$$34. u = \cot t, du = -\csc^2 t dt$$

$$\int \frac{\csc^2 t}{\cot t} dt = -\ln|\cot t| + C$$

$$35. \int \frac{\sec x \tan x}{\sec x - 1} dx = \ln|\sec x - 1| + C$$

$$36. \int (\sec t + \tan t) dt = \ln|\sec t + \tan t| - \ln|\cos t| + C$$

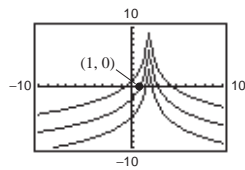
$$= \ln \left| \frac{\sec t + \tan t}{\cos t} \right| + C$$

$$= \ln|\sec t(\sec t + \tan t)| + C$$

$$37. y = \int \frac{3}{2-x} dx$$

$$= -3 \int \frac{1}{x-2} dx$$

$$= -3 \ln|x-2| + C$$



$$(1, 0): 0 = -3 \ln|1 - 2| + C \Rightarrow C = 0$$

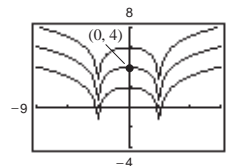
$$y = -3 \ln|x - 2|$$

$$38. y = \int \frac{2x}{x^2 - 9} dx$$

$$= \ln|x^2 - 9| + C$$

$$(0, 4): 4 = \ln|0 - 9| + C \Rightarrow C = 4 - \ln 9$$

$$y = \ln|x^2 - 9| + 4 - \ln 9$$



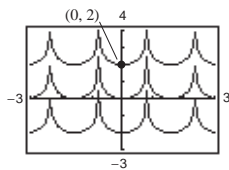
$$39. s = \int \tan(2\theta) d\theta$$

$$= \frac{1}{2} \int \tan(2\theta)(2 d\theta)$$

$$= -\frac{1}{2} \ln|\cos 2\theta| + C$$

$$(0, 2): 2 = -\frac{1}{2} \ln|\cos(0)| + C \Rightarrow C = 2$$

$$s = -\frac{1}{2} \ln|\cos 2\theta| + 2$$

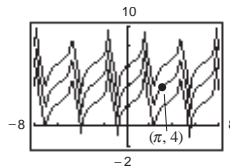


$$40. r = \int \frac{\sec^2 t}{\tan t + 1} dt$$

$$= \ln|\tan t + 1| + C$$

$$(\pi, 4): 4 = \ln|0 + 1| + C \Rightarrow C = 4$$

$$r = \ln|\tan t + 1| + 4$$



$$41. f''(x) = \frac{2}{x^2} = 2x^{-2}, \quad x > 0$$

$$f'(x) = \frac{-2}{x} + C$$

$$f'(1) = 1 = -2 + C \Rightarrow C = 3$$

$$f'(x) = \frac{-2}{x} + 3$$

$$f(x) = -2 \ln x + 3x + C_1$$

$$f(1) = 1 = -2(0) + 3 + C_1 \Rightarrow C_1 = -2$$

$$f(x) = -2 \ln x + 3x - 2$$

$$42. f''(x) = \frac{-4}{(x-1)^2} - 2 = -4(x-1)^{-2} - 2, \quad x > 1$$

$$f'(x) = \frac{4}{(x-1)} - 2x + C$$

$$f'(2) = 0 = 4 - 4 + C \Rightarrow C = 0$$

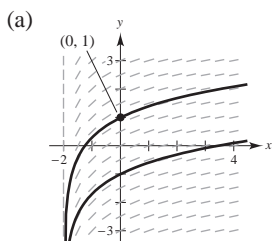
$$f'(x) = \frac{4}{x-1} - 2x$$

$$f(x) = 4 \ln(x-1) - x^2 + C_1$$

$$f(2) = 3 = 4(0) - 4 + C_1 \Rightarrow C_1 = 7$$

$$f(x) = 4 \ln(x-1) - x^2 + 7$$

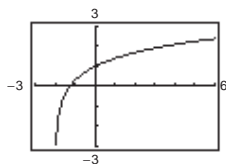
$$43. \frac{dy}{dx} = \frac{1}{x+2}, \quad (0, 1)$$



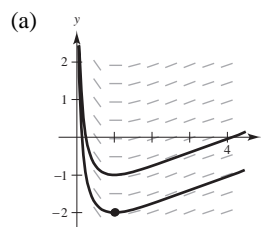
$$(b) \quad y = \int \frac{1}{x+2} dx = \ln|x+2| + C$$

$$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$$

$$\text{Hence, } y = \ln|x+2| + 1 - \ln 2 = \ln \left| \frac{x+2}{2} \right| + 1.$$



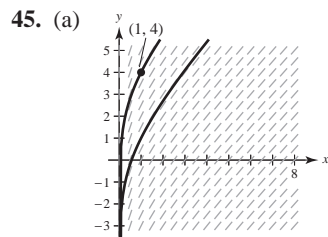
$$44. \frac{dy}{dx} = \frac{\ln x}{x}, \quad (1, -2)$$



$$(b) \quad y = \int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + C$$

$$y(1) = -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \Rightarrow C = -2$$

$$\text{Hence, } y = \frac{(\ln x)^2}{2} - 2.$$

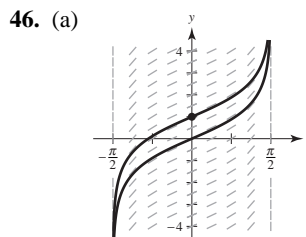
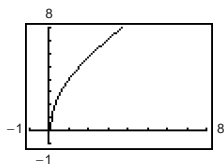


(b) $\frac{dy}{dx} = 1 + \frac{1}{x}, \quad (1, 4)$

$$y = x + \ln x + C$$

$$4 = 1 + 0 + C \Rightarrow C = 3$$

$$y = x + \ln x + 3$$

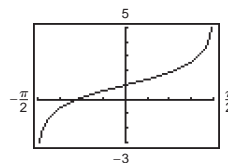


(b) $\frac{dy}{dx} = \sec x, \quad (0, 1)$

$$y = \ln|\sec x + \tan x| + C$$

$$1 = \ln|1 + 0| + C \Rightarrow C = 1$$

$$y = \ln|\sec x + \tan x| + 1$$



47. $\int_0^4 \frac{5}{3x+1} dx = \left[\frac{5}{3} \ln|3x+1| \right]_0^4$
 $= \frac{5}{3} \ln 13 \approx 4.275$

48. $\int_{-1}^1 \frac{1}{x+2} dx = \left[\ln|x+2| \right]_{-1}^1$
 $= \ln 3 - \ln 1 = \ln 3$

49. $u = 1 + \ln x, \quad du = \frac{1}{x} dx$
 $\int_1^e \frac{(1 + \ln x)^2}{x} dx = \left[\frac{1}{3} (1 + \ln x)^3 \right]_1^e$
 $= \frac{7}{3}$

50. $u = \ln x, \quad du = \frac{1}{x} dx$
 $\int_e^{e^2} \frac{1}{x \ln x} dx = \int_e^{e^2} \left(\frac{1}{\ln x} \right) \frac{1}{x} dx = \left[\ln|\ln|x|| \right]_e^{e^2} = \ln 2$

51. $\int_0^2 \frac{x^2 - 2}{x + 1} dx = \int_0^2 \left(x - 1 - \frac{1}{x + 1} \right) dx$
 $= \left[\frac{1}{2} x^2 - x - \ln|x + 1| \right]_0^2 = -\ln 3$

52. $\int_0^1 \frac{x-1}{x+1} dx = \int_0^1 1 dx + \int_0^1 \frac{-2}{x+1} dx$
 $= \left[x - 2 \ln|x+1| \right]_0^1 = 1 - 2 \ln 2$

53. $\int_1^2 \frac{1 - \cos \theta}{\theta - \sin \theta} d\theta = \left[\ln|\theta - \sin \theta| \right]_1^2$
 $= \ln \left| \frac{2 - \sin 2}{1 - \sin 1} \right| \approx 1.929$

54. $\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2 d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2 \csc 2\theta \cot 2\theta + \cot^2 2\theta) d\theta$
 $= \int_{0.1}^{0.2} (2 \csc^2 2\theta - 2 \csc 2\theta \cot 2\theta - 1) d\theta$
 $= \left[-\cot 2\theta + \csc 2\theta - \theta \right]_{0.1}^{0.2} \approx 0.0024$

55. $\int \frac{1}{1 + \sqrt{x}} dx = 2(1 + \sqrt{x}) - 2 \ln(1 + \sqrt{x}) + C_1$
 $= 2[\sqrt{x} - \ln(1 + \sqrt{x})] + C \text{ where } C = C_1 + 2.$

$$56. \int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx = -(1 + \sqrt{x})^2 + 6(1 + \sqrt{x}) - 4 \ln(1 + \sqrt{x}) + C_1$$

$$= 4\sqrt{x} - x - 4 \ln(1 + \sqrt{x}) + C \text{ where } C = C_1 + 5.$$

$$57. \int \frac{\sqrt{x}}{x-1} dx = \ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + 2\sqrt{x} + C$$

$$58. \int \frac{x^2}{x-1} dx = \ln|x-1| + \frac{x^2}{2} + x + C$$

$$59. \int_{\pi/4}^{\pi/2} (\csc x - \sin x) dx = \left[-\ln|\csc x + \cot x| + \cos x \right]_{\pi/4}^{\pi/2}$$

$$= \ln(\sqrt{2} + 1) - \frac{\sqrt{2}}{2} \approx 0.174$$

$$60. \int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x} dx = \left[\ln|\sec x + \tan x| - 2 \sin x \right]_{-\pi/4}^{\pi/4}$$

$$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$$

Note: In Exercises 61–64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

$$61. F(x) = \int_1^x \frac{1}{t} dt$$

$$F'(x) = \frac{1}{x}$$

$$62. F(x) = \int_0^x \tan t dt$$

$$F'(x) = \tan x$$

$$63. F(x) = \int_1^{3x} \frac{1}{t} dt$$

$$F'(x) = \frac{1}{3x}(3) = \frac{1}{x}$$

$$64. F(x) = \int_1^{x^2} \frac{1}{t} dt$$

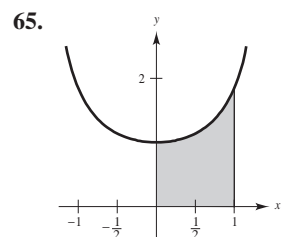
$$F'(x) = \frac{2x}{x^2} = \frac{2}{x}$$

(by Second Fundamental Theorem of Calculus)

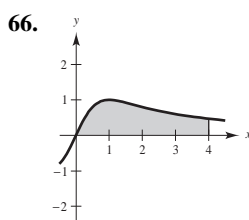
Alternate Solution:

$$F(x) = \int_1^{3x} \frac{1}{t} dt = \left[\ln|t| \right]_1^{3x} = \ln|3x|$$

$$F'(x) = \frac{1}{3x}(3) = \frac{1}{x}$$



$A \approx 1.25$; Matches (d)



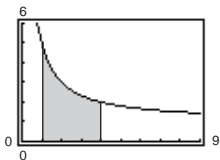
$A \approx 3$; Matches (a)

$$67. A = \int_1^3 \frac{4}{x} dx = 4 \ln|x| \Big|_1^3 = 4 \ln 3$$

$$\begin{aligned}
 68. A &= \int_2^4 \frac{2}{x \ln x} dx = 2 \int_2^4 \frac{1}{\ln x} \frac{1}{x} dx \\
 &= 2 \ln|\ln x| \Big|_2^4 \\
 &= 2[\ln(\ln 4) - \ln(\ln 2)] \\
 &= 2 \ln\left(\frac{2 \ln 2}{\ln 2}\right) = 2 \ln 2
 \end{aligned}$$

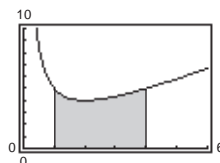
$$\begin{aligned}
 70. A &= \int_{\pi/4}^{3\pi/4} \frac{\sin x}{1 + \cos x} dx \\
 &= -\ln|1 + \cos x| \Big|_{\pi/4}^{3\pi/4} \\
 &= -\ln\left(1 - \frac{\sqrt{2}}{2}\right) + \ln\left(1 + \frac{\sqrt{2}}{2}\right) \\
 &= \ln\left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right) \\
 &= \ln(3 + 2\sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 72. A &= \int_1^4 \frac{x+4}{x} dx = \int_1^4 \left(1 + \frac{4}{x}\right) dx \\
 &= \left[x + 4 \ln x\right]_1^4 \\
 &= 4 + 4 \ln 4 - 1 \\
 &= 3 + 4 \ln 4 \approx 8.5452
 \end{aligned}$$

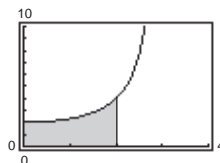


$$\begin{aligned}
 69. A &= \int_0^{\pi/4} \tan x dx = -\ln|\cos x| \Big|_0^{\pi/4} \\
 &= -\ln \frac{\sqrt{2}}{2} + 0 \\
 &= \ln \sqrt{2} = \frac{\ln 2}{2} \approx 0.3466
 \end{aligned}$$

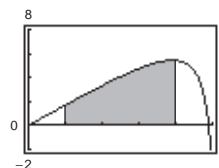
$$\begin{aligned}
 71. A &= \int_1^4 \frac{x^2 + 4}{x} dx = \int_1^4 \left(x + \frac{4}{x}\right) dx \\
 &= \left[\frac{x^2}{2} + 4 \ln x\right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2} \\
 &= \frac{15}{2} + 8 \ln 2 \approx 13.045 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 73. \int_0^2 2 \sec \frac{\pi x}{6} dx &= \frac{12}{\pi} \int_0^2 \sec\left(\frac{\pi x}{6}\right) \frac{\pi}{6} dx \\
 &= \left[\frac{12}{\pi} \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right|\right]_0^2 \\
 &= \frac{12}{\pi} \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \frac{12}{\pi} \ln|1 + 0| \\
 &= \frac{12}{\pi} \ln(2 + \sqrt{3}) \approx 5.03041
 \end{aligned}$$



$$\begin{aligned}
 74. \int_1^4 (2x - \tan(0.3x)) dx &= \left[x^2 + \frac{10}{3} \ln|\cos(0.3x)|\right]_1^4 \\
 &= \left[16 + \frac{10}{3} \ln \cos(1.2)\right] - \left[1 + \frac{10}{3} \ln \cos(0.3)\right] \approx 11.7686
 \end{aligned}$$



75. $f(x) = \frac{12}{x}$, $b - a = 5 - 1 = 4$, $n = 4$

Trapezoid: $\frac{4}{2(4)}[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] = \frac{1}{2}[12 + 12 + 8 + 6 + 2.4] = 20.2$

Simpson: $\frac{4}{3(4)}[f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)] = \frac{1}{3}[12 + 24 + 8 + 12 + 2.4] \approx 19.4667$

Calculator: $\int_1^5 \frac{12}{x} dx \approx 19.3133$

Exact: $12 \ln 5$

76. $f(x) = \frac{8x}{x^2 + 4}$, $b - a = 4 - 0 = 4$, $n = 4$

Trapezoid: $\frac{4}{2(4)}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] = \frac{1}{2}[0 + 3.2 + 4 + 3.6923 + 1.6] \approx 6.2462$

Simpson: $\frac{4}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx 6.4615$

Calculator: $\int_0^4 \frac{8x}{x^2 + 4} dx \approx 6.438$

Exact: $4 \ln 5$

77. $f(x) = \ln x$, $b - a = 6 - 2 = 4$, $n = 4$

Trapezoid: $\frac{4}{2(4)}[f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)] = \frac{1}{2}[0.6931 + 2.1972 + 2.7726 + 3.2189 + 1.7918] \approx 5.3368$

Simpson: $\frac{4}{3(4)}[f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)] \approx 5.3632$

Calculator: $\int_2^6 \ln x dx \approx 5.3643$

78. $f(x) = \sec x$, $b - a = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$, $n = 4$

Trapezoid: $\frac{2\pi/3}{2(4)}\left[f\left(-\frac{\pi}{3}\right) + 2f\left(-\frac{\pi}{6}\right) + 2f(0) + 2f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right)\right] \approx \frac{\pi}{12}[2 + 2.3094 + 2 + 2.3094 + 2] \approx 2.780$

Simpson: $\frac{2\pi/3}{3(4)}\left[f\left(-\frac{\pi}{3}\right) + 4f\left(-\frac{\pi}{6}\right) + 2f(0) + 4f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 2.6595$

Calculator: $\int_{-\pi/3}^{\pi/3} \sec x dx \approx 2.6339$

79. Power Rule

80. Substitution: ($u = x^2 + 4$) and Power Rule

81. Substitution: ($u = x^2 + 4$) and Log Rule

82. Substitution: ($u = \tan x$) and Log Rule

$$\begin{aligned} 83. -\ln|\cos x| + C &= \ln\left|\frac{1}{\cos x}\right| + C \\ &= \ln|\sec x| + C \end{aligned}$$

$$\begin{aligned} 84. \ln|\sin x| + C &= \ln\left|\frac{1}{\csc x}\right| + C \\ &= -\ln|\csc x| + C \end{aligned}$$

$$\begin{aligned}
 85. \ln|\sec x + \tan x| + C &= \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C \\
 &= \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C
 \end{aligned}$$

$$\begin{aligned}
 86. -\ln|\csc x + \cot x| + C &= -\ln\left|\frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C \\
 &= -\ln\left|\frac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C
 \end{aligned}$$

$$\begin{aligned}
 87. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{8}{x^2} dx \\
 &= 4 \int_2^4 x^{-2} dx \\
 &= \left[-4 \frac{1}{x}\right]_2^4 \\
 &= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1
 \end{aligned}$$

$$\begin{aligned}
 88. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{4(x+1)}{x^2} dx \\
 &= 2 \int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx \\
 &= 2 \left[\ln x - \frac{1}{x}\right]_2^4 \\
 &= 2 \left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right] \\
 &= 2 \left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863
 \end{aligned}$$

$$\begin{aligned}
 89. \text{Average value} &= \frac{1}{e-1} \int_1^e \frac{\ln x}{x} dx \\
 &= \frac{1}{e-1} \left[\frac{(\ln x)^2}{2}\right]_1^e \\
 &= \frac{1}{e-1} \left(\frac{1}{2}\right) \\
 &= \frac{1}{2e-2} \approx 0.291
 \end{aligned}$$

$$\begin{aligned}
 90. \text{Average value} &= \frac{1}{2-0} \int_0^2 \sec \frac{\pi x}{6} dx \\
 &= \left[\frac{1}{2} \left(\frac{6}{\pi}\right) \ln \left|\sec \frac{\pi x}{6} + \tan \frac{\pi x}{6}\right|\right]_0^2 \\
 &= \frac{3}{\pi} [\ln(2 + \sqrt{3}) - \ln(1 + 0)] \\
 &= \frac{3}{\pi} \ln(2 + \sqrt{3})
 \end{aligned}$$

$$\begin{aligned}
 91. P(t) &= \int \frac{3000}{1+0.25t} dt = (3000)(4) \int \frac{0.25}{1+0.25t} dt \\
 &= 12,000 \ln|1+0.25t| + C \\
 P(0) &= 12,000 \ln|1+0.25(0)| + C = 1000 \\
 C &= 1000 \\
 P(t) &= 12,000 \ln|1+0.25t| + 1000 \\
 &= 1000[12 \ln|1+0.25t| + 1] \\
 P(3) &= 1000[12(\ln 1.75) + 1] \approx 7715
 \end{aligned}$$

$$\begin{aligned}
 92. t &= \frac{10}{\ln 2} \int_{250}^{300} \frac{1}{T-100} dT \\
 &= \frac{10}{\ln 2} \left[\ln(T-100)\right]_{250}^{300} = \frac{10}{\ln 2} [\ln 200 - \ln 150] \\
 &= \frac{10}{\ln 2} \left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504 \text{ units of time}
 \end{aligned}$$

$$\begin{aligned}
 93. \frac{1}{50-40} \int_{40}^{50} \frac{90,000}{400+3x} dx &= \left[3000 \ln|400+3x|\right]_{40}^{50} \\
 &\approx \$168.27
 \end{aligned}$$

$$94. \frac{dS}{dt} = \frac{k}{t}$$

$$S(t) = \int \frac{k}{t} dt = k \ln|t| + C = k \ln t + C \text{ since } t > 1.$$

$$S(2) = k \ln 2 + C = 200$$

$$S(4) = k \ln 4 + C = 300$$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

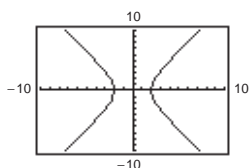
$$S(t) = \frac{100 \ln t}{\ln 2} + 100 = 100 \left[\frac{\ln t}{\ln 2} + 1 \right].$$

$$95. (a) 2x^2 - y^2 = 8$$

$$y^2 = 2x^2 - 8$$

$$y_1 = \sqrt{2x^2 - 8}$$

$$y_2 = -\sqrt{2x^2 - 8}$$



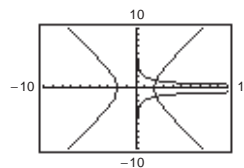
$$(c) \text{ In part (a): } 2x^2 - y^2 = 8$$

$$4x - 2yy' = 0$$

$$y' = \frac{2x}{y}$$

$$(b) y^2 = e^{-f(1/x)} dx = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \frac{1}{x}k$$

$$\text{Let } k = 4 \text{ and graph } y^2 = \frac{4}{x}. \quad \left(y_1 = \frac{2}{\sqrt{x}}, y_2 = -\frac{2}{\sqrt{x}} \right)$$



$$\text{In part (b): } y^2 = \frac{4}{x} = 4x^{-1}$$

$$2yy' = \frac{-4}{x^2}$$

$$y' = \frac{-2}{yx^2} = \frac{-2y}{y^2x^2} = \frac{-2y}{4x} = \frac{-y}{2x}$$

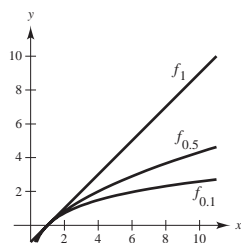
Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

$$96. k = 1: f_1(x) = x - 1$$

$$k = 0.5: f_{0.5}(x) = \frac{\sqrt{x} - 1}{0.5} = 2(\sqrt{x} - 1)$$

$$k = 0.1: f_{0.1}(x) = \frac{10\sqrt{x} - 1}{0.1} = 10(10\sqrt{x} - 1)$$

$$\lim_{k \rightarrow 0^+} f_k(x) = \ln x$$



97. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \\ \neq (\ln x)^{1/2}$$

98. False

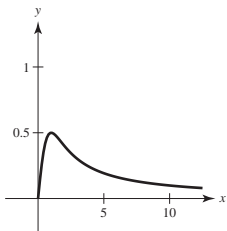
$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

99. True

$$\int \frac{1}{x} dx = \ln|x| + C_1 \\ = \ln|x| + \ln|C| \\ = \ln|Cx|, C \neq 0$$

100. False; the integrand has a nonremovable discontinuity at $x = 0$.

101. $f(x) = \frac{x}{1+x^2}$



(a) $y = \frac{1}{2}x$ intersects $f(x) = \frac{x}{1+x^2}$:

$$\frac{1}{2}x = \frac{x}{1+x^2}$$

$$1+x^2 = 2$$

$$x = 1$$

$$A = \int_0^1 \left(\left[\frac{x}{1+x^2} \right] - \frac{1}{2}x \right) dx$$

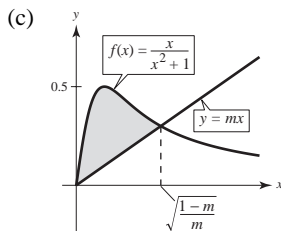
$$= \left[\frac{1}{2} \ln(x^2 + 1) - \frac{x^2}{4} \right]_0^1$$

$$= \frac{1}{2} \ln 2 - \frac{1}{4}$$

(b) $f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

$$f'(0) = 1$$

Hence, for $0 < m < 1$, the graphs of f and $y = mx$ enclose a finite region.



$$\frac{x}{1+x^2} = mx$$

$$1 = m + mx^2$$

$$x^2 = \frac{1-m}{m}$$

$$x = \sqrt{\frac{1-m}{m}}, \text{ intersection point}$$

$$A = \int_0^{\sqrt{(1-m)/m}} \left(\frac{x}{1+x^2} - mx \right) dx, \quad 0 < m < 1$$

$$= \left[\frac{1}{2} \ln(1+x^2) - \frac{mx^2}{2} \right]_0^{\sqrt{(1-m)/m}}$$

$$= \frac{1}{2} \ln \left(1 + \frac{1-m}{m} \right) - \frac{1}{2} m \left(\frac{1-m}{m} \right)$$

$$= \frac{1}{2} \ln \left(\frac{1}{m} \right) - \frac{1}{2} (1-m)$$

$$= \frac{1}{2} [m - \ln(m) - 1]$$

102. $F(x) = \int_x^{2x} \frac{1}{t} dt, \quad x > 0$

$$F'(x) = \frac{1}{2x}(2) - \frac{1}{x} = 0 \implies F \text{ is constant on } (0, \infty).$$

Alternate Solution:

$$F(x) = \ln t \Big|_x^{2x} = \ln(2x) - \ln x$$

$$= \ln 2 + \ln x - \ln x$$

$$= \ln 2$$

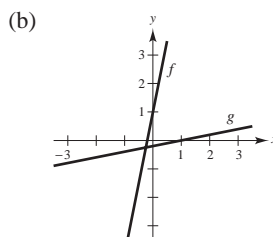
Section 5.3 Inverse Functions

1. (a) $f(x) = 5x + 1$

$$g(x) = \frac{x-1}{5}$$

$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$

$$g(f(x)) = g(5x+1) = \frac{(5x+1)-1}{5} = x$$

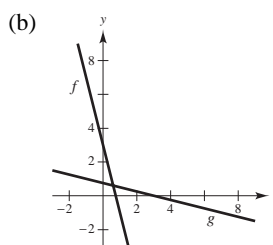


2. (a) $f(x) = 3 - 4x$

$$g(x) = \frac{3-x}{4}$$

$$f(g(x)) = f\left(\frac{3-x}{4}\right) = 3 - 4\left(\frac{3-x}{4}\right) = x$$

$$g(f(x)) = g(3-4x) = \frac{3-(3-4x)}{4} = x$$

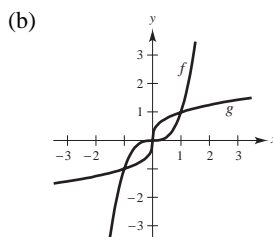


3. (a) $f(x) = x^3$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$



4. (a) $f(x) = 1 - x^3$

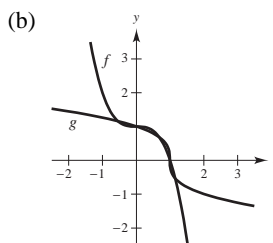
$$g(x) = \sqrt[3]{1-x}$$

$$f(g(x)) = f(\sqrt[3]{1-x}) = 1 - (\sqrt[3]{1-x})^3$$

$$= 1 - (1-x) = x$$

$$g(f(x)) = g(1-x^3)$$

$$= \sqrt[3]{1-(1-x^3)} = \sqrt[3]{x^3} = x$$



5. (a) $f(x) = \sqrt{x-4}$

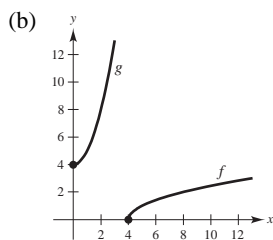
$$g(x) = x^2 + 4, \quad x \geq 0$$

$$f(g(x)) = f(x^2 + 4)$$

$$= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x$$

$$g(f(x)) = g(\sqrt{x-4})$$

$$= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x$$

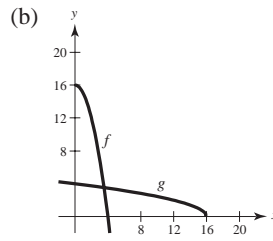


6. (a) $f(x) = 16 - x^2, \quad x \geq 0$

$$g(x) = \sqrt{16 - x}$$

$$f(g(x)) = f(\sqrt{16 - x}) = 16 - (\sqrt{16 - x})^2 = 16 - (16 - x) = x$$

$$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)} = \sqrt{x^2} = x$$

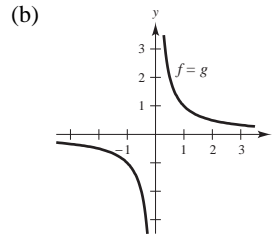


7. (a) $f(x) = \frac{1}{x}$

$$g(x) = \frac{1}{x}$$

$$f(g(x)) = \frac{1}{1/x} = x$$

$$g(f(x)) = \frac{1}{1/x} = x$$

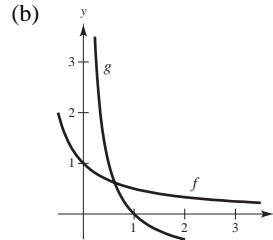


8. (a) $f(x) = \frac{1}{1 + x}, \quad x \geq 0$

$$g(x) = \frac{1 - x}{x}, \quad 0 < x \leq 1$$

$$f(g(x)) = f\left(\frac{1 - x}{x}\right) = \frac{1}{1 + \frac{1 - x}{x}} = \frac{1}{\frac{1 + 1 - x}{x}} = \frac{1}{\frac{2 - x}{x}} = \frac{x}{2 - x} = x$$

$$g(f(x)) = g\left(\frac{1}{1 + x}\right) = \frac{1 - \frac{1}{1 + x}}{\frac{1}{1 + x}} = \frac{\frac{1 + x - 1}{1 + x}}{\frac{1}{1 + x}} = \frac{x}{1 + x} \cdot \frac{1 + x}{1} = x$$



9. Matches (c)

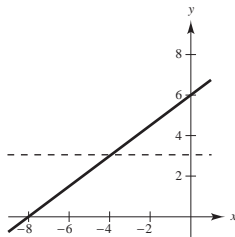
10. Matches (b)

11. Matches (a)

12. Matches (d)

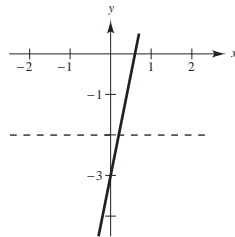
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



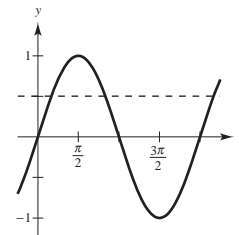
14. $f(x) = 5x - 3$

One-to-one; has an inverse



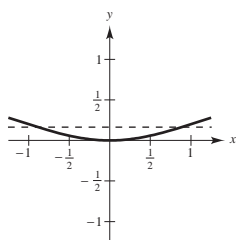
15. $f(\theta) = \sin \theta$

Not one-to-one; does not have an inverse



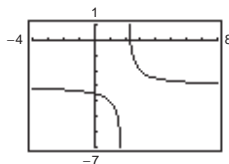
16. $F(x) = \frac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



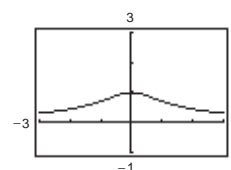
17. $h(s) = \frac{1}{s-2} - 3$

One-to-one; has an inverse



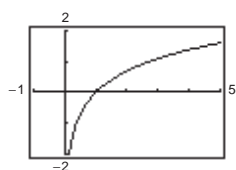
18. $g(t) = \frac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



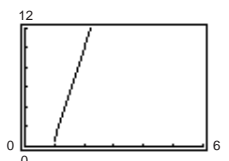
19. $f(x) = \ln x$

One-to-one; has an inverse



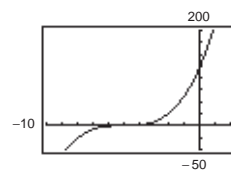
20. $f(x) = 5x\sqrt{x-1}$

One-to-one; has an inverse



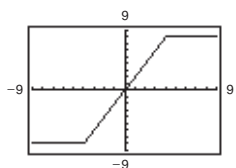
21. $g(x) = (x+5)^3$

One-to-one; has an inverse



22. $h(x) = |x+4| - |x-4|$

Not one-to-one; does not have an inverse



23. $f(x) = \ln(x-3), x > 3$

$$f'(x) = \frac{1}{x-3} > 0 \text{ for } x > 3$$

f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

24. $f(x) = \cos \frac{3x}{2}$

$$f'(x) = -\frac{3}{2} \sin \frac{3x}{2} = 0 \text{ when } x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

25. $f(x) = \frac{x^4}{4} - 2x^2$

$$f'(x) = x^3 - 4x = 0 \text{ when } x = 0, 2, -2$$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$$f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2 \geq 0 \text{ for all } x$$

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

27. $f(x) = 2 - x - x^3$

$$f'(x) = -1 - 3x^2 < 0 \text{ for all } x$$

f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = (x+a)^3 + b$

$$f'(x) = 3(x+a)^2 \geq 0 \text{ for all } x$$

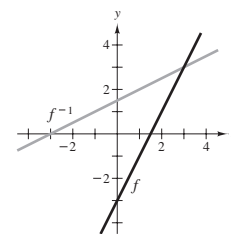
f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y+3}{2}$$

$$y = \frac{x+3}{2}$$

$$f^{-1}(x) = \frac{x+3}{2}$$

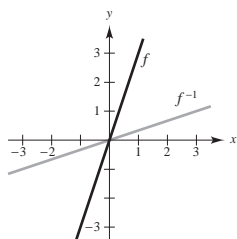


30. $f(x) = 3x = y$

$$x = \frac{y}{3}$$

$$y = \frac{x}{3}$$

$$f^{-1}(x) = \frac{x}{3}$$

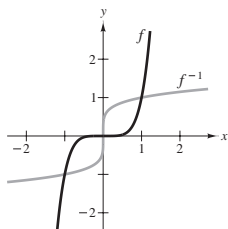


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

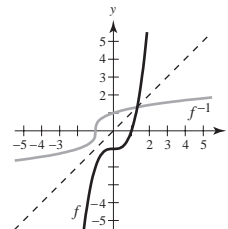


32. $f(x) = x^3 - 1 = y$

$$x = \sqrt[3]{y + 1}$$

$$y = \sqrt[3]{x + 1}$$

$$f^{-1}(x) = \sqrt[3]{x + 1}$$

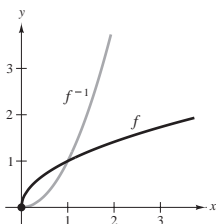


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

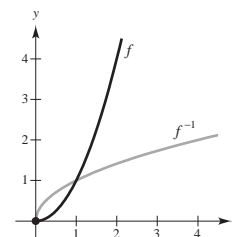


34. $f(x) = x^2 = y, \quad 0 \leq x$

$$x = \sqrt{y}$$

$$y = \sqrt{x}$$

$$f^{-1}(x) = \sqrt{x}$$

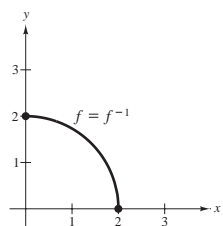


35. $f(x) = \sqrt{4 - x^2} = y, \quad 0 \leq x \leq 2$

$$x = \sqrt{4 - y^2}$$

$$y = \sqrt{4 - x^2}$$

$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

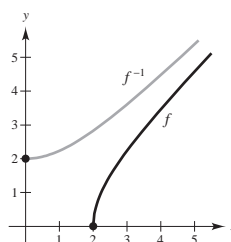


36. $f(x) = \sqrt{x^2 - 4} = y, \quad x \geq 2$

$$x = \sqrt{y^2 + 4}$$

$$y = \sqrt{x^2 + 4}$$

$$f^{-1}(x) = \sqrt{x^2 + 4}, \quad x \geq 0$$

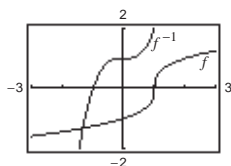


37. $f(x) = \sqrt[3]{x - 1} = y$

$$x = y^3 + 1$$

$$y = x^3 + 1$$

$$f^{-1}(x) = x^3 + 1$$



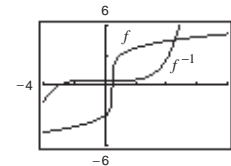
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

38. $f(x) = 3\sqrt[5]{2x - 1} = y$

$$x = \frac{y^5 + 243}{486}$$

$$y = \frac{x^5 + 243}{486}$$

$$f^{-1}(x) = \frac{x^5 + 243}{486}$$



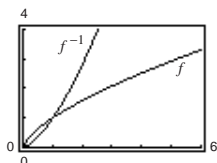
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \quad x \geq 0$

$x = y^{3/2}$

$y = x^{3/2}$

$f^{-1}(x) = x^{3/2}, \quad x \geq 0$



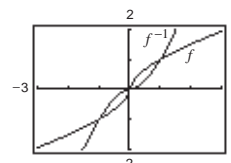
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

40. $f(x) = x^{3/5} = y$

$x = y^{5/3}$

$y = x^{5/3}$

$f^{-1}(x) = x^{5/3}$



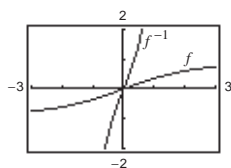
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \frac{x}{\sqrt{x^2 + 7}} = y$

$x = \frac{\sqrt{7}y}{\sqrt{1 - y^2}}$

$y = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}$

$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

42. $f(x) = \frac{x + 2}{x} = y$

$x = \frac{2}{y - 1}$

$y = \frac{2}{x - 1}$

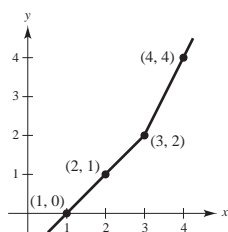
$f^{-1}(x) = \frac{2}{x - 1}$



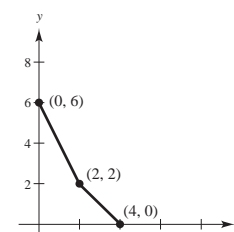
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

 43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4


 44.

x	0	2	4
$f^{-1}(x)$	6	2	0



45. (a) Let x be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$y = 1.25x + 1.60(50 - x)$$

$$= -0.35x + 80, \quad 0 \leq x \leq 50.$$

- (c) Domain of inverse is $62.5 \leq x \leq 80$.

- (b) We find the inverse of the original function:

$$y = -0.35x + 80$$

$$0.35x = 80 - y$$

$$x = \frac{100}{35}(80 - y)$$

$$\text{Inverse: } y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x)$$

x represents cost and y represents pounds.

- (d) If $x = 73$ in the inverse function,
 $y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$ pounds.

46. $C = \frac{5}{9}(F - 32), \quad F \geq -459.6$

(a) $\frac{9}{5}C = F - 32$

$$F = 32 + \frac{9}{5}C$$

- (c) For $F \geq -459.6$, $C = \frac{5}{9}(F - 32) \geq -273.\bar{1}$.
 Therefore, domain is $C \geq -273.\bar{1} = -273\frac{1}{9}$.

- (b) The inverse function gives the temperature F corresponding to the Celsius temperature C .

- (d) If $C = 22^\circ$, then $F = 32 + \frac{9}{5}(22) = 71.6^\circ\text{F}$.

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$$f'(x) = 2(x - 4) > 0 \text{ on } (4, \infty)$$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

48. $f(x) = |x + 2|$ on $[-2, \infty)$

$$f'(x) = \frac{|x + 2|}{x + 2}(1) = 1 > 0 \text{ on } (-2, \infty)$$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

49. $f(x) = \frac{4}{x^2}$ on $(0, \infty)$

$$f'(x) = -\frac{8}{x^3} < 0 \text{ on } (0, \infty)$$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$$f'(x) = -\sin x < 0 \text{ on } (0, \pi)$$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \frac{x}{x^2 - 4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} (1 - \sqrt{1 + 16x^2})/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Domain of f^{-1} : all x

Range of f^{-1} : $-2 < y < 2$

50. $f(x) = \cot x$ on $(0, \pi)$

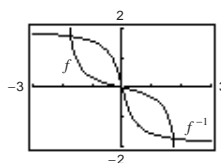
$$f'(x) = -\csc^2 x < 0 \text{ on } (0, \pi)$$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \sec x$ on $\left[0, \frac{\pi}{2}\right)$

$$f'(x) = \sec x \tan x > 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

f is increasing on $\left[0, \frac{\pi}{2}\right)$. Therefore, f is strictly monotonic and has an inverse.



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

54. $f(x) = 2 - \frac{3}{x^2} = y$ on $(0, 10)$

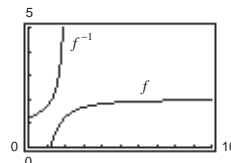
$$2x^2 - 3 = x^2y$$

$$x^2(2 - y) = 3$$

$$x = \pm \sqrt{\frac{3}{2 - y}}$$

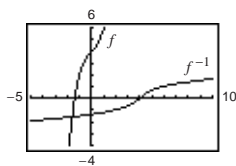
$$y = \pm \sqrt{\frac{3}{2 - x}}$$

$$f^{-1}(x) = \sqrt{\frac{3}{2 - x}}, \quad x < 2$$



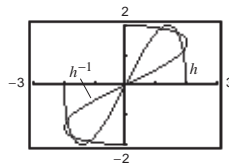
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



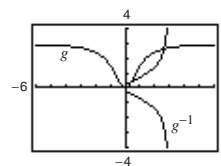
(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

56. (a), (b)



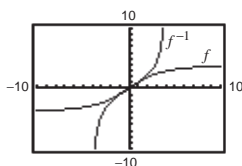
(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

60. $f(x) = -3$

Not one-to-one; does not have an inverse

61. $f(x) = |x - 2|$, $x \leq 2$

$$= -(x - 2)$$

$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$

$$2 - y = x$$

$$f^{-1}(x) = 2 - x, \quad x \geq 0$$

63. $f(x) = (x - 3)^2$ is one-to-one for $x \geq 3$.

$$(x - 3)^2 = y$$

$$x - 3 = \sqrt{y}$$

$$x = \sqrt{y} + 3$$

$$y = \sqrt{x} + 3$$

$$f^{-1}(x) = \sqrt{x} + 3, \quad x \geq 0$$

(Answer is not unique.)

65. $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$

$$x = y - 3$$

$$y = x + 3$$

$$f^{-1}(x) = x - 3, \quad x \geq 0$$

(Answer is not unique.)

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V .

59. $f(x) = \sqrt{x - 2}$, Domain: $x \geq 2$

$$f'(x) = \frac{1}{2\sqrt{x - 2}} > 0 \text{ for } x > 2$$

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2$$

$$x = y^2 + 2$$

$$y = x^2 + 2$$

$$f^{-1}(x) = x^2 + 2, \quad x \geq 0$$

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$$ax + b = y$$

$$x = \frac{y - b}{a}$$

$$y = \frac{x - b}{a}$$

$$f^{-1}(x) = \frac{x - b}{a}, \quad a \neq 0$$

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$$16 - x^4 = y$$

$$16 - y = x^4$$

$$\sqrt[4]{16 - y} = x$$

$$\sqrt[4]{16 - x} = y$$

$$f^{-1}(x) = \sqrt[4]{16 - x}, \quad x \leq 16$$

66. $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$$x - 3 = y$$

$$x = y + 3$$

$$y = x + 3$$

$$f^{-1}(x) = x + 3, \quad x \geq 0$$

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

$$71. \quad f(x) = x^3 + 2x - 1, f(1) = 2 = a$$

$$f'(x) = 3x^2 + 2$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

$$73. \quad f(x) = \sin x, f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$$

$$f'(x) = \cos x$$

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)}$$

$$= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

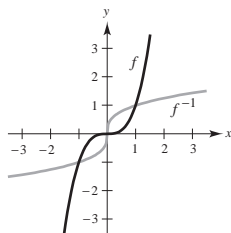
$$75. \quad f(x) = x^3 - \frac{4}{x}, f(2) = 6 = a$$

$$f'(x) = 3x^2 + \frac{4}{x^2}$$

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

77. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(c)



70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A .

$$72. \quad f(x) = \frac{1}{27}(x^5 + 2x^3)$$

$$f(-3) = \frac{1}{27}(-243 - 54) = -11 = a$$

$$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$$

$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))}$$

$$= \frac{1}{f'(-3)} = \frac{27}{5(-3)^4 + 6(-3)^2} = \frac{1}{17}$$

$$74. \quad f(x) = \cos 2x, f(0) = 1 = a$$

$$f'(x) = -2 \sin 2x$$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)}$$

$$= \frac{1}{-2 \sin 0} = \frac{1}{0} \text{ which is undefined.}$$

$$76. \quad f(x) = \sqrt{x-4}, f(8) = 2 = a$$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)}$$

$$= \frac{1}{1/(2\sqrt{8-4})} = \frac{1}{1/4} = 4$$

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$

$$(d) \quad f(x) = x^3, \left(\frac{1}{2}, \frac{1}{8}\right)$$

$$f'(x) = 3x^2$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$

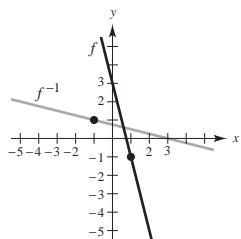
$$f^{-1}(x) = \sqrt[3]{x}, \left(\frac{1}{8}, \frac{1}{2}\right)$$

$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

78. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(c)

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$

(d) $f(x) = 3 - 4x, (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

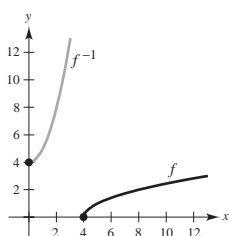
$f^{-1}(x) = \frac{3-x}{4}, (-1, 1)$

$(f^{-1})'(x) = -\frac{1}{4}$

$(f^{-1})'(-1) = -\frac{1}{4}$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(c)

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(d) $f(x) = \sqrt{x-4}, (5, 1)$

$f'(x) = \frac{1}{2\sqrt{x-4}}$

$f'(5) = \frac{1}{2}$

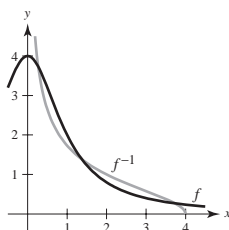
$f^{-1}(x) = x^2 + 4, (1, 5)$

$(f^{-1})'(x) = 2x$

$(f^{-1})'(1) = 2$

80. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(c)

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(d) $f(x) = \frac{4}{1+x^2}$

$f'(x) = \frac{-8x}{(x^2+1)^2}, f'(1) = -2$

$f^{-1}(x) = \sqrt{\frac{4-x}{x}}$

$(f^{-1})'(x) = \frac{-2}{x^2\sqrt{(4-x)/x}}, (f^{-1})'(2) = -\frac{1}{2}$

81. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2 \frac{dy}{dx} - 14y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}$$

$$\text{At } (-4, 1), \frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}$$

Alternate Solution:Let $f(x) = x^3 - 7x^2 + 2$. Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$. Hence,

$$\frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}$$

82. $x = 2 \ln(y^2 - 3)$

$$1 = 2 \frac{1}{y^2 - 3} 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2 - 3}{4y}$$

$$\text{At } (0, 4), \frac{dy}{dx} = \frac{16 - 3}{16} = \frac{13}{16}$$

In Exercises 83–86, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x + 3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

$$83. (f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$$

$$84. (g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3)) = g^{-1}(0) = 0$$

$$85. (f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$$

$$86. (g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}(\sqrt[3]{-4}) \\ = \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[2]{4}$$

In Exercises 87–90, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

$$87. (g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x)) \\ = g^{-1}(x - 4) \\ = \frac{(x - 4) + 5}{2} \\ = \frac{x + 1}{2}$$

$$88. (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) \\ = f^{-1}\left(\frac{x + 5}{2}\right) \\ = \frac{x + 5}{2} - 4 \\ = \frac{x - 3}{2}$$

$$89. (f \circ g)(x) = f(g(x)) \\ = f(2x - 5) \\ = (2x - 5) + 4 \\ = 2x - 1$$

$$\text{Hence, } (f \circ g)^{-1}(x) = \frac{x + 1}{2}.$$

$$\text{Note: } (f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

$$90. (g \circ f)(x) = g(f(x)) \\ = g(x + 4) \\ = 2(x + 4) - 5 \\ = 2x + 3$$

$$\text{Hence, } (g \circ f)^{-1}(x) = \frac{x - 3}{2}.$$

$$\text{Note: } (g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

91. Answers will vary. See page 343 and Example 3.

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

93. f is not one-to-one because many different x -values yield the same y -value.

$$\text{Example: } f(0) = f(\pi) = 0$$

Not continuous at $\frac{(2n - 1)\pi}{2}$, where n is an integer.

94. f is not one-to-one because different x -values yield the same y -value.

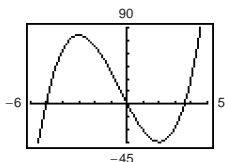
$$\text{Example: } f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$$

Not continuous at ± 2 .

95. $f(x) = k(2 - x - x^3)$ is one-to-one. Since $f^{-1}(3) = -2$,
 $f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \Rightarrow k = \frac{1}{4}$.

96. (a) $f(x) = 2x^3 + 3x^2 - 36x$

f does not pass the horizontal line test.



(b) $f'(x) = 6x^2 + 6x - 36$

$$= 6(x^2 + x - 6) = 6(x + 3)(x - 2)$$

$$f'(x) = 0 \text{ at } x = 2, -3$$

Hence, on the interval $[-2, 2]$, f is one-to-one.

97. Let f and g be one-to-one functions.

(a) Let $(f \circ g)(x_1) = (f \circ g)(x_2)$

$$f(g(x_1)) = f(g(x_2))$$

$$g(x_1) = g(x_2) \quad (\text{Because } f \text{ is one-to-one.})$$

$$x_1 = x_2 \quad (\text{Because } g \text{ is one-to-one.})$$

Thus, $f \circ g$ is one-to-one.

(b) Let $(f \circ g)(x) = y$, then $x = (f \circ g)^{-1}(y)$. Also:

$$(f \circ g)(x) = y$$

$$f(g(x)) = y$$

$$g(x) = f^{-1}(y)$$

$$x = g^{-1}(f^{-1}(y))$$

$$x = (g^{-1} \circ f^{-1})(y)$$

Thus, $(f \circ g)^{-1}(y) = (g^{-1} \circ f^{-1})(y)$ and

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let

$$(f^{-1})^{-1}(x) = y \text{ then } x = f^{-1}(y) \text{ and } f(x) = y.$$

$$\text{Thus, } (f^{-1})^{-1} = f.$$

99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a) . Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

101. False. Let $f(x) = x^2$.

102. True; if f has a y -intercept

103. True

104. False. Let $f(x) = x$ or $g(x) = 1/x$.

105. Not true.

$$\text{Let } f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1 - x, & 1 < x \leq 2 \end{cases}$$

f is one-to-one, but not strictly monotonic.

106. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/(f'(g(x)))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

107. $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}, f(2) = 0$

$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

$$108. f(x) = \int_2^x \sqrt{1+t^2} dt, f(2) = 0$$

$$f'(x) = \sqrt{1+x^2}$$

$f'(x) > 0$ for all $x \Rightarrow f$ increasing on $(-\infty, \infty) \Rightarrow f$ is one-to-one.

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$109. (a) \quad y = \frac{x-2}{x-1}$$

$$x = \frac{y-2}{y-1}$$

$$xy - x = y - 2$$

$$xy - y = x - 2$$

$$y = \frac{x-2}{x-1}$$

Hence, if $f(x) = \frac{x-2}{x-1}$, then $f^{-1}(x) = f(x)$.

(b) The graph of f is symmetric about the line $y = x$.

Section 5.4 Exponential Functions: Differentiation and Integration

$$1. e^{\ln x} = 4$$

$$x = 4$$

$$2. e^{\ln 2x} = 12$$

$$2x = 12$$

$$x = 6$$

$$3. e^x = 12$$

$$x = \ln 12 \approx 2.485$$

$$4. 4e^x = 83$$

$$e^x = \frac{83}{4}$$

$$x = \ln\left(\frac{83}{4}\right) \approx 3.033$$

$$5. 9 - 2e^x = 7$$

$$2e^x = 2$$

$$e^x = 1$$

$$x = 0$$

$$6. -6 + 3e^x = 8$$

$$3e^x = 14$$

$$e^x = \frac{14}{3}$$

$$x = \ln\left(\frac{14}{3}\right)$$

$$\approx 1.540$$

$$7. 50e^{-x} = 30$$

$$e^{-x} = \frac{3}{5}$$

$$-x = \ln\left(\frac{3}{5}\right)$$

$$x = \ln\left(\frac{5}{3}\right)$$

$$\approx 0.511$$

$$8. 200e^{-4x} = 15$$

$$e^{-4x} = \frac{15}{200} = \frac{3}{40}$$

$$-4x = \ln\left(\frac{3}{40}\right)$$

$$x = \frac{1}{4} \ln\left(\frac{40}{3}\right)$$

$$\approx 0.648$$

$$9. \ln x = 2$$

$$x = e^2 \approx 7.389$$

$$10. \ln x^2 = 10$$

$$x^2 = e^{10}$$

$$x = \pm e^5 \approx \pm 148.4132$$

$$11. \ln(x-3) = 2$$

$$x-3 = e^2$$

$$x = 3 + e^2 \approx 10.389$$

$$12. \ln 4x = 1$$

$$4x = e^1 = e$$

$$x = \frac{e}{4} \approx 0.680$$

$$13. \ln\sqrt{x+2} = 1$$

$$\sqrt{x+2} = e^1 = e$$

$$x+2 = e^2$$

$$x = e^2 - 2 \approx 5.389$$

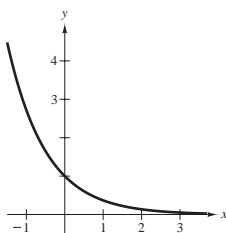
$$14. \ln(x-2)^2 = 12$$

$$(x-2)^2 = e^{12}$$

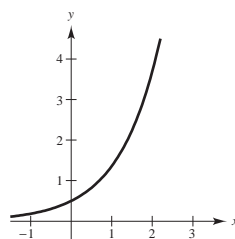
$$x-2 = e^6$$

$$x = 2 + e^6 \approx 405.429$$

$$15. y = e^{-x}$$

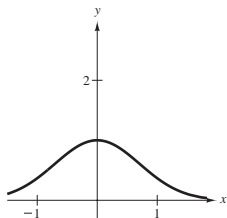


$$16. y = \frac{1}{2}e^x$$

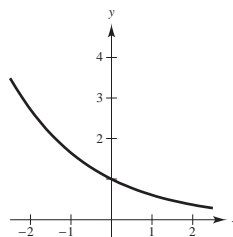


17. $y = e^{-x^2}$

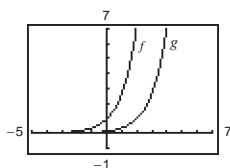
Symmetric with respect to the y-axis

 Horizontal asymptote: $y = 0$


18. $y = e^{-x/2}$

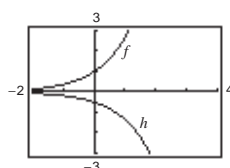


19. (a)



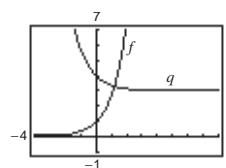
Horizontal shift 2 units to the right

(b)



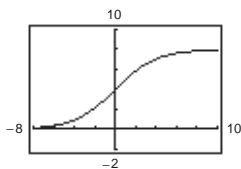
A reflection in the x-axis and a vertical shrink

(c)

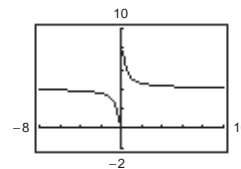


Vertical shift 3 units upward and a reflection in the y-axis

20. (a)


 Horizontal asymptotes: $y = 0$ and $y = 8$

(b)


 Horizontal asymptote: $y = 4$

21. $y = Ce^{ax}$

 Horizontal asymptote: $y = 0$

Matches (c)

22. $y = Ce^{-ax}$

 Horizontal asymptote: $y = 0$

Reflection in the y-axis

Matches (d)

23. $y = C(1 - e^{-ax})$

 Vertical shift C units

Reflection in both the x- and y-axes

Matches (a)

24. $y = \frac{C}{1 + e^{-ax}}$

$$\lim_{x \rightarrow \infty} \frac{C}{1 + e^{-ax}} = C$$

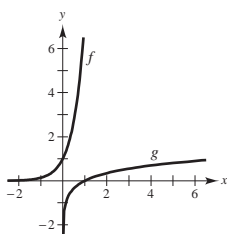
$$\lim_{x \rightarrow -\infty} \frac{C}{1 + e^{-ax}} = 0$$

 Horizontal asymptotes:
 $y = C$ and $y = 0$

Matches (b)

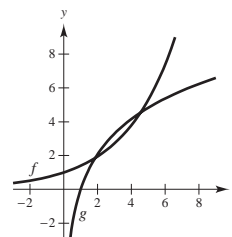
25. $f(x) = e^{2x}$

$$g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$



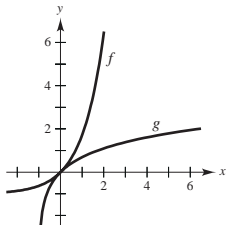
26. $f(x) = e^{x/3}$

$$g(x) = \ln x^3 = 3 \ln x$$



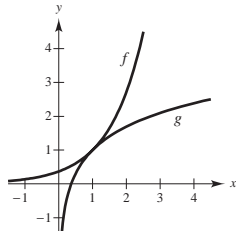
27. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$

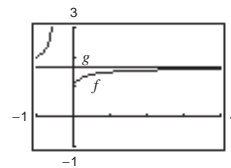


28. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



29.



As $x \rightarrow \infty$, the graph of f approaches the graph of g .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

30. In the same way,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

31. $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000} \approx 2.718280469$

$e \approx 2.718281828$

$e > \left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$

32. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 2.71825396$

$e \approx 2.718281828$

$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

33. (a) $y = e^{3x}$

$y' = 3e^{3x}$

$y'(0) = 3$

$y - 1 = 3(x - 0)$

$y = 3x + 1$ Tangent line

(b) $y = e^{-3x}$

$y' = -3e^{-3x}$

$y'(0) = -3$

$y - 1 = -3(x - 0)$

$y = -3x + 1$ Tangent line

34. (a) $y = e^{2x}$

$y' = 2e^{2x}$

$y'(0) = 2$

$y - 1 = 2(x - 0)$

$y = 2x + 1$

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

$y'(0) = -2$

$y - 1 = -2(x - 0)$

$y = -2x + 1$

35. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

36. $y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

37. $y = e^{\sqrt{x}}$

$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

38. $y = x^2e^{-x}$

$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x}$

$= xe^{-x}(2 - x)$

39. $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^{-t} - e^t)$

40. $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \frac{6}{t^3e^{3/t^2}}$

41. $y = \ln(1 + e^{2x})$

$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$

42. $y = \ln\left(\frac{1 + e^x}{1 - e^x}\right) = \ln(1 + e^x) - \ln(1 - e^x)$

$\frac{dy}{dx} = \frac{e^x}{1 + e^x} + \frac{e^x}{1 - e^x} = \frac{2e^x}{1 - e^{2x}}$

$$43. y = \frac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$$

$$\frac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$45. y = e^x(\sin x + \cos x)$$

$$\begin{aligned} \frac{dy}{dx} &= e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x) \\ &= e^x(2 \cos x) = 2e^x \cos x \end{aligned}$$

$$47. F(x) = \int_{\pi}^{\ln x} \cos e^t dt$$

$$F'(x) = \cos(e^{\ln x}) \cdot \frac{1}{x} = \frac{\cos(x)}{x}$$

$$49. f(x) = e^{1-x}, \quad (1, 1)$$

$$f'(x) = -e^{1-x}, \quad f'(1) = -1$$

$$y - 1 = -1(x - 1)$$

$$y = -x + 2 \quad \text{Tangent line}$$

$$51. y = \ln(e^{x^2}) = x^2, \quad (-2, 4)$$

$$y' = 2x, \quad y'(-2) = -4$$

$$y - 4 = -4(x + 2)$$

$$y = -4x - 4 \quad \text{Tangent line}$$

$$53. y = x^2e^x - 2xe^x + 2e^x, \quad (1, e)$$

$$y' = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x$$

$$y'(1) = e$$

$$y - e = e(x - 1)$$

$$y = ex \quad \text{Tangent line}$$

$$55. f(x) = e^{-x} \ln x, \quad (1, 0)$$

$$f'(x) = e^{-x} \left(\frac{1}{x} \right) - e^{-x} \ln x = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

$$f'(1) = e^{-1}$$

$$y - 0 = e^{-1}(x - 1)$$

$$y = \frac{1}{e}x - \frac{1}{e} \quad \text{Tangent line}$$

$$44. y = \frac{e^x - e^{-x}}{2}$$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$$

$$46. y = \ln e^x = x$$

$$\frac{dy}{dx} = 1$$

$$48. F(x) = \int_0^{e^{2x}} \ln(t + 1) dt$$

$$F'(x) = \ln(e^{2x} + 1)2e^{2x} = 2e^{2x} \ln(e^{2x} + 1)$$

$$50. y = e^{-2x+x^2}, \quad (2, 1)$$

$$y' = (2x - 2)e^{-2x+x^2}, \quad y'(2) = 2$$

$$y - 1 = 2(x - 2)$$

$$y = 2x - 3 \quad \text{Tangent line}$$

$$52. y = \ln \frac{e^x + e^{-x}}{2}, \quad (0, 0)$$

$$y' = \frac{1}{[(e^x + e^{-x})/2]} [e^x - e^{-x}]$$

$$y'(0) = 0$$

$$y = 0 \quad \text{Tangent line}$$

$$54. y = xe^x - e^x, \quad (1, 0)$$

$$y' = xe^x + e^x - e^x = xe^x$$

$$y'(1) = e$$

$$y - 0 = e(x - 1)$$

$$y = ex - e \quad \text{Tangent line}$$

$$56. f(x) = e^3 \ln x, \quad (1, 0)$$

$$f'(x) = \frac{e^3}{x}, \quad f'(1) = e^3$$

$$y - 0 = e^3(x - 1)$$

$$y = e^3(x - 1) \quad \text{Tangent line}$$

57. $xe^y - 10x + 3y = 0$

$$xe^y \frac{dy}{dx} + e^y - 10 + 3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

58. $e^{xy} + x^2 - y^2 = 10$

$$\left(x \frac{dy}{dx} + y\right)e^{xy} + 2x - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = \frac{-ye^{xy} + 2x}{xe^{xy} - 2y}$$

59. $xe^y + ye^x = 1, (0, 1)$

$$xe^y y' + e^y + ye^x + y'e^x = 0$$

$$\text{At } (0, 1): e + 1 + y' = 0$$

$$y' = -e - 1$$

$$\text{Tangent line: } y - 1 = (-e - 1)(x - 0)$$

$$y = (-e - 1)x + 1$$

60. $1 + \ln(xy) = e^{x-y}, (1, 1)$

$$\frac{1}{xy}[xy' + y] = e^{x-y}[1 - y']$$

$$\text{At } (1, 1): [y' + 1] = 1 - y'$$

$$y' = 0$$

$$\text{Tangent line: } y - 1 = 0(x - 1)$$

$$y = 1$$

61. $f(x) = (3 + 2x)e^{-3x}$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

62. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$

$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$$

$$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x$$

63. $y = e^x(\cos \sqrt{2}x + \sin \sqrt{2}x)$

$$y' = e^x(-\sqrt{2} \sin \sqrt{2}x + \sqrt{2} \cos \sqrt{2}x) + e^x(\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$= e^x[(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$y'' = e^x[-(\sqrt{2} + 2) \sin \sqrt{2}x + (\sqrt{2} - 2) \cos \sqrt{2}x] + e^x[(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$= e^x[(-1 - 2\sqrt{2}) \sin \sqrt{2}x + (-1 + 2\sqrt{2}) \cos \sqrt{2}x]$$

$$-2y' + 3y = -2e^x[(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x] + 3e^x[\cos \sqrt{2}x + \sin \sqrt{2}x]$$

$$= e^x[(1 - 2\sqrt{2}) \cos \sqrt{2}x + (1 + 2\sqrt{2}) \sin \sqrt{2}x] = -y''$$

$$\text{Therefore, } -2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0.$$

64. $y = e^x(3 \cos 2x - 4 \sin 2x)$

$$y' = e^x(-6 \sin 2x - 8 \cos 2x) + e^x(3 \cos 2x - 4 \sin 2x)$$

$$= e^x(-10 \sin 2x - 5 \cos 2x) = -5e^x(2 \sin 2x + \cos 2x)$$

$$y'' = -5e^x(4 \cos 2x - 2 \sin 2x) - 5e^x(2 \sin 2x + \cos 2x) = -5e^x(5 \cos 2x) = -25e^x \cos 2x$$

$$y'' - 2y' = -25e^x \cos 2x - 2(-5e^x)(2 \sin 2x + \cos 2x) = -5e^x(3 \cos 2x - 4 \sin 2x) = -5y$$

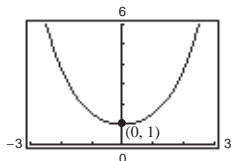
$$\text{Therefore, } y'' - 2y' = -5y \Rightarrow y'' - 2y' + 5y = 0.$$

$$65. f(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{e^x + e^{-x}}{2} > 0$$

Relative minimum: (0, 1)

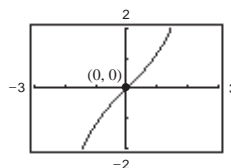


$$66. f(x) = \frac{e^x - e^{-x}}{2}$$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: (0, 0)



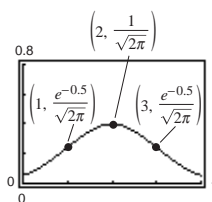
$$67. g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$$

$$g'(x) = \frac{-1}{\sqrt{2\pi}} (x-2) e^{-(x-2)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}} (x-1)(x-3) e^{-(x-2)^2/2}$$

Relative maximum: $\left(2, \frac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



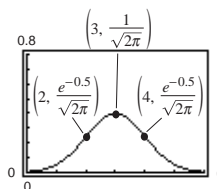
$$68. g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}$$

$$g'(x) = \frac{-1}{\sqrt{2\pi}} (x-3) e^{-(x-3)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}} (x-2)(x-4) e^{-(x-3)^2/2}$$

Relative maximum: $\left(3, \frac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection: $\left(2, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right), \left(4, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$



$$69. f(x) = x^2 e^{-x}$$

$$f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x} (2-x) = 0 \text{ when } x = 0, 2.$$

$$f''(x) = -e^{-x} (2x - x^2) + e^{-x} (2 - 2x) \\ = e^{-x} (x^2 - 4x + 2) = 0 \text{ when } x = 2 \pm \sqrt{2}.$$

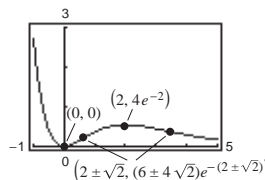
Relative minimum: (0, 0)

Relative maximum: $(2, 4e^{-2})$

$$x = 2 \pm \sqrt{2}$$

$$y = (2 \pm \sqrt{2})^2 e^{-(2 \pm \sqrt{2})}$$

Points of inflection: (3.414, 0.384), (0.586, 0.191)

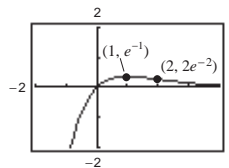


70. $f(x) = xe^{-x}$

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) = 0 \text{ when } x = 1.$$

$$f''(x) = -e^{-x} + (-e^{-x})(1 - x) = e^{-x}(x - 2) = 0 \text{ when } x = 2.$$

 Relative maximum: $(1, e^{-1})$

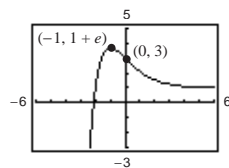
 Point of inflection: $(2, 2e^{-2})$


71. $g(t) = 1 + (2 + t)e^{-t}$

$$g'(t) = -(1 + t)e^{-t}$$

$$g''(t) = te^{-t}$$

 Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

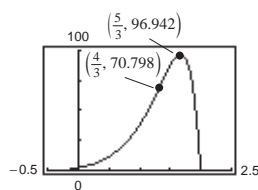
 Point of inflection: $(0, 3)$


72. $f(x) = -2 + e^{3x}(4 - 2x)$

$$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0 \text{ when } x = \frac{5}{3}.$$

$$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0 \text{ when } x = \frac{4}{3}.$$

 Relative maximum: $(\frac{5}{3}, 96.942)$

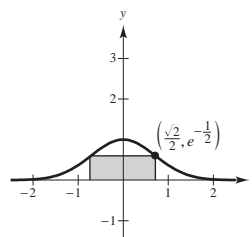
 Point of inflection: $(\frac{4}{3}, 70.798)$


73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

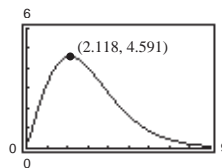
$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

$$(b) A(x) = xf(c) = x \left[10 \left(\frac{x}{e^x - 1} \right) e^{-x/(e^x - 1)} \right]$$

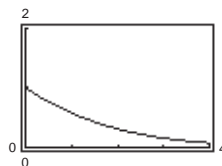
$$= \frac{10x^2}{e^x - 1} e^{x/(1 - e^x)}$$

(c) $A(x) = \frac{10x^2}{e^x - 1} e^{x/(1 - e^x)}$



The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \frac{x}{e^x - 1}$



$$\lim_{x \rightarrow 0^+} c = 1$$

$$\lim_{x \rightarrow \infty} c = 0$$

75. $y = \frac{L}{1 + ae^{-x/b}}, \quad a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

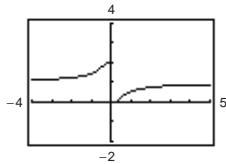
$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

76. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞ , $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

77. $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

Let $(x, y) = (x, e^{2x})$ be the point on the graph where the tangent line passes through the origin. Equating slopes,

$$2e^{2x} = \frac{e^{2x} - 0}{x - 0}$$

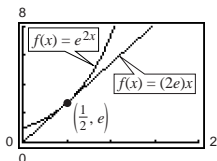
$$2 = \frac{1}{x}$$

$$x = \frac{1}{2}, y = e, y' = 2e.$$

Point: $\left(\frac{1}{2}, e\right)$

$$\text{Tangent line: } y - e = 2e\left(x - \frac{1}{2}\right)$$

$$y = 2ex$$



78. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad (\text{Slope of tangent line})$$

$$-\frac{1}{y'} = e^x \quad (\text{Slope of normal line})$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$

We want $(0, 0)$ to satisfy the equation:

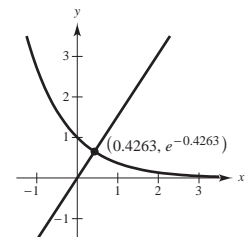
$$-e^{-x_0} = -x_0e^{x_0}$$

$$1 = x_0e^{2x_0}$$

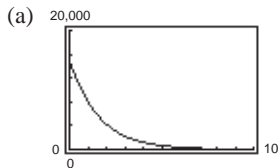
$$x_0e^{2x_0} - 1 = 0$$

Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$$(0.4263, e^{-0.4263})$$



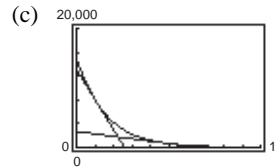
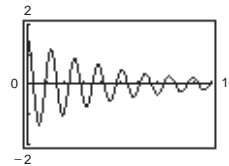
79. $V = 15,000e^{-0.6286t}$, $0 \leq t \leq 10$



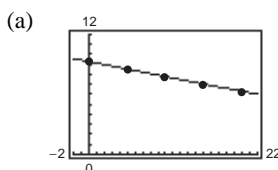
(b) $\frac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1$, $\frac{dV}{dt} \approx -5028.84$.

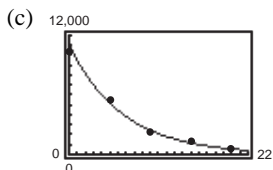
When $t = 5$, $\frac{dV}{dt} \approx -406.89$.


 80. $1.56e^{-0.22t} \cos 4.9t \leq 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \geq 7.79$ seconds.

 81.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.



(b) $\ln P = ah + b$

$$P = e^{ah+b} = e^b e^{ah}$$

$$P = C e^{ah}, C = e^b$$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10,957.7$.

$$P = 10,957.7e^{-0.1499h}$$

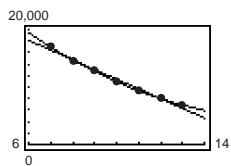
(d) $\frac{dP}{dh} = (10,957.71)(-0.1499)e^{-0.1499h}$

$$= -1642.56e^{-0.1499h}$$

For $h = 5$, $\frac{dP}{dh} = -776.3$. For $h = 18$, $\frac{dP}{dh} \approx -110.6$.

 82. (a) Linear model: $V = -1686.8t + 28,242$

Quadratic model: $V = 109.52t^2 - 3877.3t + 38,756$



(b) The slope represents the average loss in value per year.

(c) Exponential model: $V = 49,591.06(0.8592)^t$
 $= 49,591.06e^{-0.1518t}$

 (d) As $t \rightarrow \infty$, $V \rightarrow 0$ for the exponential model. The value tends to zero.

 (e) When $t = 8$, $V' \approx -2235$ dollars/year.

When $t = 12$, $V' \approx -1218$ dollars/year.

$$83. \quad f(x) = e^{x/2}, \quad f(0) = 1$$

$$f'(x) = \frac{1}{2}e^{x/2}, \quad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{x/2}, \quad f''(0) = \frac{1}{4}$$

$$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, \quad P_1(0) = 1$$

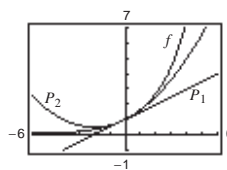
$$P_1'(x) = \frac{1}{2}, \quad P_1'(0) = \frac{1}{2}$$

$$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 \quad P_2(0) = 1$$

$$= \frac{x^2}{8} + \frac{x}{2} + 1$$

$$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, \quad P_2'(0) = \frac{1}{2}$$

$$P_2''(x) = \frac{1}{4}, \quad P_2''(0) = \frac{1}{4}$$



The values of f , P_1 , P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

$$84. \quad f(x) = e^{-x^2/2}, \quad f(0) = 1$$

$$f'(x) = -xe^{-x^2/2}, \quad f'(0) = 0$$

$$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), \quad f''(0) = -1$$

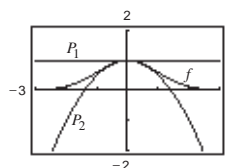
$$P_1(x) = 1 + 0(x - 0) = 1, \quad P_1(0) = 1$$

$$P_1'(x) = 0, \quad P_1'(0) = 0$$

$$P_2(x) = 1 + 0(x - 0) - \frac{1}{2}(x - 0)^2 = 1 - \frac{x^2}{2}, \quad P_2(0) = 1$$

$$P_2'(x) = -x, \quad P_2'(0) = 0$$

$$P_2''(x) = -1, \quad P_2''(0) = -1$$



The values of f , P_1 , P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

$$85. \quad \text{Let } u = 5x, \quad du = 5 \, dx.$$

$$\int e^{5x}(5) \, dx = e^{5x} + C$$

$$86. \quad \text{Let } u = -x^4, \quad du = -4x^3 \, dx.$$

$$\int e^{-x^4}(-4x^3) \, dx = e^{-x^4} + C$$

$$87. \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = 2 \int e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2e^{\sqrt{x}} + C$$

$$88. \quad \int \frac{e^{1/x^2}}{x^3} \, dx = -\frac{1}{2} \int e^{1/x^2} \left(\frac{-2}{x^3} \right) dx = -\frac{1}{2}e^{1/x^2} + C$$

$$89. \quad \text{Let } u = 1 + e^{-x}, \quad du = -e^{-x} \, dx.$$

$$\int \frac{e^{-x}}{1 + e^{-x}} \, dx = - \int \frac{-e^{-x}}{1 + e^{-x}} \, dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

90. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \ln(1 + e^{2x}) + C$$

92. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x}) dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln|e^x + e^{-x}| + C$$

94. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x}) dx$.

$$\begin{aligned} \int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx &= 2 \int (e^x + e^{-x})^{-2} (e^x - e^{-x}) dx \\ &= \frac{-2}{e^x + e^{-x}} + C \end{aligned}$$

96.
$$\int \frac{e^{2x} + 2e^x + 1}{e^x} dx = \int (e^x + 2 + e^{-x}) dx = e^x + 2x - e^{-x} + C$$

98.
$$\int \ln(e^{2x-1}) dx = \int (2x - 1) dx = x^2 - x + C$$

100.
$$\int_3^4 e^{3-x} dx = \left[-e^{3-x} \right]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$$

101.
$$\begin{aligned} \int_0^1 xe^{-x^2} dx &= -\frac{1}{2} \int_0^1 e^{-x^2} (-2x) dx \\ &= -\frac{1}{2} \left[e^{-x^2} \right]_0^1 \\ &= -\frac{1}{2} [e^{-1} - 1] \\ &= \frac{1 - (1/e)}{2} = \frac{e - 1}{2e} \end{aligned}$$

103. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2} dx$.

$$\begin{aligned} \int_1^3 \frac{e^{3/x}}{x^2} dx &= -\frac{1}{3} \int_1^3 e^{3/x} \left(-\frac{3}{x^2} \right) dx \\ &= \left[-\frac{1}{3} e^{3/x} \right]_1^3 = \frac{e}{3} (e^2 - 1) \end{aligned}$$

91. Let $u = 1 - e^x$, $du = -e^x dx$.

$$\begin{aligned} \int e^x \sqrt{1 - e^x} dx &= - \int (1 - e^x)^{1/2} (-e^x) dx \\ &= -\frac{2}{3} (1 - e^x)^{3/2} + C \end{aligned}$$

93. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x}) dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln|e^x - e^{-x}| + C$$

95.
$$\begin{aligned} \int \frac{5 - e^x}{e^{2x}} dx &= \int 5e^{-2x} dx - \int e^{-x} dx \\ &= -\frac{5}{2} e^{-2x} + e^{-x} + C \end{aligned}$$

97.
$$\begin{aligned} \int e^{-x} \tan(e^{-x}) dx &= - \int [\tan(e^{-x})] (-e^{-x}) dx \\ &= \ln|\cos(e^{-x})| + C \end{aligned}$$

99. Let $u = -2x$, $du = -2 dx$.

$$\begin{aligned} \int_0^1 e^{-2x} dx &= -\frac{1}{2} \int_0^1 e^{-2x} (-2) dx = \left[-\frac{1}{2} e^{-2x} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-2}) = \frac{e^2 - 1}{2e^2} \end{aligned}$$

102.
$$\begin{aligned} \int_{-2}^0 x^2 e^{x^3/2} dx &= \frac{2}{3} \int_{-2}^0 e^{x^3/2} \left(\frac{3}{2} x^2 \right) dx \\ &= \frac{2}{3} \left[e^{x^3/2} \right]_{-2}^0 \\ &= \frac{2}{3} [1 - e^{-4}] \\ &= \frac{2}{3} \left[1 - \frac{1}{e^4} \right] = \frac{2(e^4 - 1)}{3e^4} \end{aligned}$$

104. Let $u = \frac{-x^2}{2}$, $du = -x dx$.

$$\begin{aligned} \int_0^{\sqrt{2}} xe^{-x^2/2} dx &= - \int_0^{\sqrt{2}} e^{-x^2/2} (-x) dx \\ &= \left[-e^{-x^2/2} \right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e - 1}{e} \end{aligned}$$

$$\begin{aligned}
 105. \int_0^{\pi/2} e^{\sin \pi x} \cos \pi x \, dx &= \frac{1}{\pi} \int_0^{\pi/2} e^{\sin \pi x} (\pi \cos \pi x) \, dx \\
 &= \frac{1}{\pi} \left[e^{\sin \pi x} \right]_0^{\pi/2} \\
 &= \frac{1}{\pi} [e^{\sin(\pi^2/2)} - 1]
 \end{aligned}$$

$$\begin{aligned}
 106. \int_{\pi/3}^{\pi/2} e^{\sec 2x} \sec 2x \tan 2x \, dx &= \frac{1}{2} \int_{\pi/3}^{\pi/2} e^{\sec 2x} (2 \sec 2x \tan 2x) \, dx \\
 &= \frac{1}{2} \left[e^{\sec 2x} \right]_{\pi/3}^{\pi/2} \\
 &= \frac{1}{2} [e^{-1} - e^{-2}] \\
 &= \frac{1}{2} \left[\frac{1}{e} - \frac{1}{e^2} \right] = \frac{e-1}{2e^2}
 \end{aligned}$$

107. Let $u = ax^2$, $du = 2ax \, dx$. (Assume $a \neq 0$.)

$$\begin{aligned}
 y &= \int x e^{ax^2} \, dx \\
 &= \frac{1}{2a} \int e^{ax^2} (2ax) \, dx = \frac{1}{2a} e^{ax^2} + C
 \end{aligned}$$

$$109. f'(x) = \int \frac{1}{2}(e^x + e^{-x}) \, dx = \frac{1}{2}(e^x - e^{-x}) + C_1$$

$$f'(0) = C_1 = 0$$

$$f(x) = \int \frac{1}{2}(e^x - e^{-x}) \, dx = \frac{1}{2}(e^x + e^{-x}) + C_2$$

$$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$$

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\begin{aligned}
 108. y &= \int (e^x - e^{-x})^2 \, dx \\
 &= \int (e^{2x} - 2 + e^{-2x}) \, dx \\
 &= \frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} + C
 \end{aligned}$$

$$110. f'(x) = \int (\sin x + e^{2x}) \, dx = -\cos x + \frac{1}{2} e^{2x} + C_1$$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2} e^{2x} + 1$$

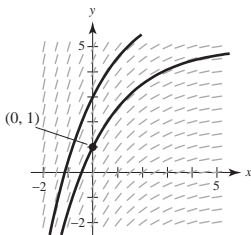
$$f(x) = \int \left(-\cos x + \frac{1}{2} e^{2x} + 1 \right) \, dx$$

$$= -\sin x + \frac{1}{4} e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \Rightarrow C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4} e^{2x}$$

111. (a)



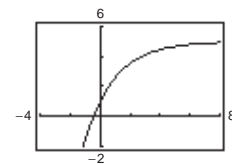
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2} \, dx = -4 \int e^{-x/2} \left(-\frac{1}{2} \, dx \right)$$

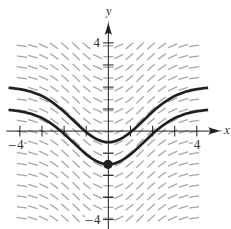
$$= -4e^{-x/2} + C$$

$$(0, 1): 1 = -4e^0 + C = -4 + C \Rightarrow C = 5$$

$$y = -4e^{-x/2} + 5$$



112. (a)



(b) $\frac{dy}{dx} = xe^{-0.2x^2}, \left(0, -\frac{3}{2}\right)$

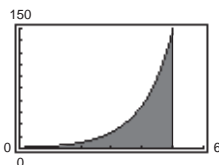
$$y = \int xe^{-0.2x^2} dx = \frac{1}{-0.4} \int e^{-0.2x^2} (-0.4x) dx$$

$$= -\frac{1}{0.4} e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

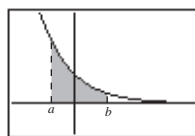
$$\left(0, -\frac{3}{2}\right): -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \Rightarrow C = 1$$

$$y = -2.5e^{-0.2x^2} + 1$$

113. $\int_0^5 e^x dx = \left[e^x \right]_0^5 = e^5 - 1 \approx 147.413$

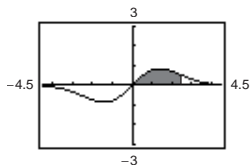


114. $\int_a^b e^{-x} dx = \left[-e^{-x} \right]_a^b = e^{-a} - e^{-b}$



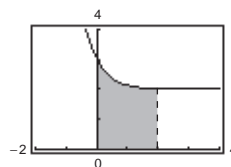
115. $\int_0^{\sqrt{6}} xe^{-x^2/4} dx = \left[-2e^{-x^2/4} \right]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$



116. $\int_0^2 (e^{-2x} + 2) dx = \left[-\frac{1}{2}e^{-2x} + 2x \right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



117. $\int_0^4 \sqrt{x}e^x dx, n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing utility: 92.7437

118. $\int_0^2 2xe^{-x} dx, n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing utility: 1.18799

119. $0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2} dt$

 Graphing utility: $0.4772 = 47.72\%$

120. $\int_0^x 0.3^{-0.3t} dt = \frac{1}{2}$

$$\left[-e^{-0.3t} \right]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln \frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

$$121. \int_0^x e^t dt \geq \int_0^x 1 dt$$

$$\left[e^t \right]_0^x \geq \left[t \right]_0^x$$

$$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x \text{ for } x \geq 0$$

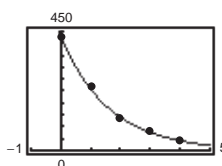
122.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

$$(a) \ln R = -0.6155t + 6.0609$$

$$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$$

(b)



$$(c) \int_0^4 R(t) dt = \int_0^4 428.78e^{-0.6155t} dt \approx 637.2 \text{ liters}$$

123. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$$\lim_{x \rightarrow -\infty} e^x = 0 \text{ and } \lim_{x \rightarrow \infty} e^x = \infty.$$

124. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

125. Yes. $f(x) = Ce^x$, C a constant.

126. (a) Log Rule: ($u = e^x + 1$)

(b) Substitution: ($u = x^2$)

$$127. e^{-x} = x \Rightarrow f(x) = x - e^{-x}$$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

We approximate the root of f to be $x = 0.567$.

$$128. \text{Area} = \frac{8}{3} = \int_{-a}^a e^{-x} dx = -e^{-x} \Big|_{-a}^a = -e^{-a} + e^a$$

Let $z = e^a$:

$$\frac{8}{3} = \frac{-1}{z} + z$$

$$\frac{8}{3}z = -1 + z^2$$

$$3z^2 - 8z - 3 = 0$$

$$(3z + 1)(z - 3) = 0$$

$$z = 3 \Rightarrow e^a = 3 \Rightarrow a = \ln 3$$

$$\left(z = -\frac{1}{3} \Rightarrow e^a = -\frac{1}{3} \text{ impossible} \right)$$

Answer: $a = \ln 3$

$$129. \ln \frac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$$

$$\ln e^{a-b} = a - b$$

Therefore, $\ln \frac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\frac{e^a}{e^b} = e^{a-b}$.

130. $f(x) = \frac{\ln x}{x}$

(a) $f'(x) = \frac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e), f'(x) > 0 \Rightarrow f$ is increasing.

On $(e, \infty), f'(x) < 0 \Rightarrow f$ is decreasing.

(b) For $e \leq A < B$, we have

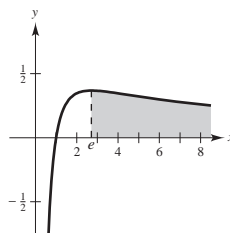
$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.



Section 5.5 Bases Other than e and Applications

1. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

2. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

3. $\log_7 1 = 0$

4. $\log_a \frac{1}{a} = \log_a 1 - \log_a a = -1$

5. (a) $2^3 = 8$

6. (a) $27^{2/3} = 9$

$\log_2 8 = 3$

$\log_{27} 9 = \frac{2}{3}$

(b) $3^{-1} = \frac{1}{3}$

(b) $16^{3/4} = 8$

$\log_3 \frac{1}{3} = -1$

$\log_{16} 8 = \frac{3}{4}$

7. (a) $\log_{10} 0.01 = -2$

(b) $\log_{0.5} 8 = -3$

8. (a) $\log_3 \frac{1}{9} = -2$

(b) $49^{1/2} = 7$

$10^{-2} = 0.01$

$0.5^{-3} = 8$

$3^{-2} = \frac{1}{9}$

$\log_{49} 7 = \frac{1}{2}$

$(\frac{1}{2})^{-3} = 8$

9. $y = 3^x$

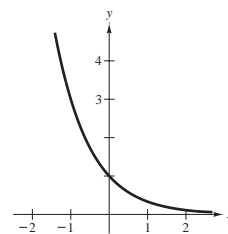
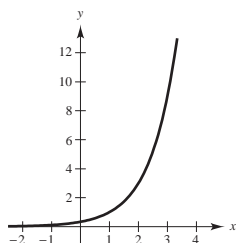
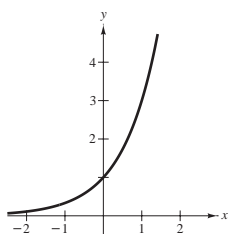
10. $y = 3^{x-1}$

11. $y = (\frac{1}{3})^x = 3^{-x}$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

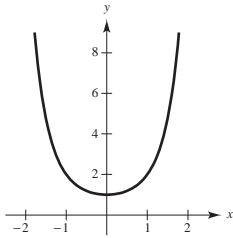
x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



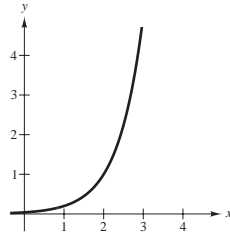
12. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



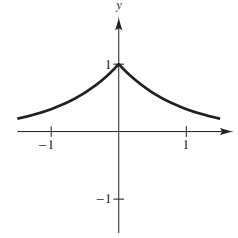
13. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



14. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



15. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$

$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$

$$x = -1$$

16. (a) $\log_3 \frac{1}{81} = x$

$$3^x = \frac{1}{81}$$

$$x = -4$$

(b) $\log_6 36 = x$

$$6^x = 36$$

$$x = 2$$

17. (a) $\log_3 x = -1$

$$3^{-1} = x$$

$$x = \frac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$

$$x = \frac{1}{16}$$

18. (a) $\log_b 27 = 3$

$$b^3 = 27$$

$$b = 3$$

(b) $\log_b 125 = 3$

$$b^3 = 125$$

$$b = 5$$

19. (a) $x^2 - x = \log_5 25$

$$x^2 - x = \log_5 5^2 = 2$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \text{ OR } x = 2$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$

$$3x = 1$$

$$x = \frac{1}{3}$$

20. (a) $\log_3 x + \log_3(x - 2) = 1$

$$\log_3[x(x - 2)] = 1$$

$$x(x - 2) = 3^1$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x + 3) - \log_{10} x = 1$

$$\log_{10} \frac{x + 3}{x} = 1$$

$$\frac{x + 3}{x} = 10^1$$

$$x + 3 = 10x$$

$$3 = 9x$$

$$x = \frac{1}{3}$$

21. $3^{2x} = 75$

$$2x \ln 3 = \ln 75$$

$$x = \left(\frac{1}{2}\right) \frac{\ln 75}{\ln 3} \approx 1.965$$

22. $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$

$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

23. $2^{3-z} = 625$
 $(3 - z) \ln 2 = \ln 625$

$$3 - z = \frac{\ln 625}{\ln 2}$$

$$z = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

24. $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$

$$(x - 1) \ln 5 = \ln\left(\frac{86}{3}\right)$$

$$x - 1 = \frac{\ln(86/3)}{\ln 5}$$

$$x = 1 + \frac{\ln(86/3)}{\ln 5} \approx 3.085$$

25. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$

$$t = \left(\frac{1}{12}\right) \frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

26. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$

$$t = \left(\frac{1}{365}\right) \frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

27. $\log_2(x - 1) = 5$

$$x - 1 = 2^5 = 32$$

$$x = 33$$

28. $\log_{10}(t - 3) = 2.6$

$$t - 3 = 10^{2.6}$$

$$t = 3 + 10^{2.6} \approx 401.107$$

29. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$

$$x = \pm \sqrt{3^{4.5}} \approx \pm 11.845$$

30. $\log_5 \sqrt{x - 4} = 3.2$

$$\sqrt{x - 4} = 5^{3.2}$$

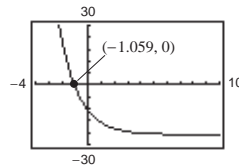
$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$

$$x = 4 + 5^{6.4}$$

$$\approx 29,748.593$$

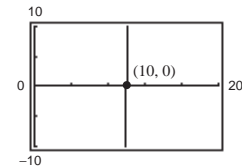
31. $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$



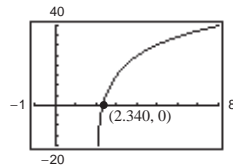
32. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



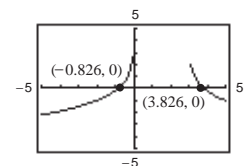
33. $h(s) = 32 \log_{10}(s - 2) + 15$

Zero: $s \approx 2.340$



34. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

Zeros: $x \approx -0.826, 3.826$

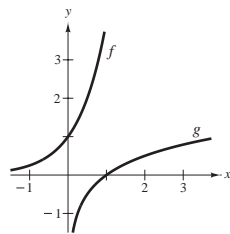


35. $f(x) = 4^x$

$$g(x) = \log_4 x$$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1

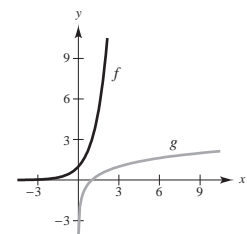


36. $f(x) = 3^x$

$$g(x) = \log_3 x$$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



$$37. f(x) = 4^x$$

$$f'(x) = (\ln 4)4^x$$

$$38. y = x(6^{-2x})$$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}] + 6^{-2x}$$

$$= 6^{-2x}[-2x(\ln 6) + 1]$$

$$= 6^{-2x}(1 - 2x \ln 6)$$

$$39. g(t) = t^2 2^t$$

$$g'(t) = t^2(\ln 2)2^t + (2t)2^t$$

$$= t^2(t \ln 2 + 2)$$

$$= 2^t t(2 + t \ln 2)$$

$$40. f(t) = \frac{3^{2t}}{t}$$

$$f'(t) = \frac{t(2 \ln 3)3^{2t} - 3^{2t}}{t^2}$$

$$= \frac{3^{2t}(2t \ln 3 - 1)}{t^2}$$

$$41. h(\theta) = 2^{-\theta} \cos \pi \theta$$

$$h'(\theta) = 2^{-\theta}(-\pi \sin \pi \theta) - (\ln 2)2^{-\theta} \cos \pi \theta$$

$$= -2^{-\theta}[(\ln 2) \cos \pi \theta + \pi \sin \pi \theta]$$

$$42. g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$$

$$g'(\alpha) = 5^{-\alpha/2} 2 \cos 2\alpha - \frac{1}{2}(\ln 5)5^{-\alpha/2} \sin 2\alpha$$

$$43. f(x) = \log_2 \frac{x^2}{x-1}$$

$$= 2 \log_2 x - \log_2(x-1)$$

$$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$$

$$= \frac{x-2}{(\ln 2)x(x-1)}$$

$$44. h(x) = \log_3 \frac{x\sqrt{x-1}}{2}$$

$$= \log_3 x + \frac{1}{2} \log_3(x-1) - \log_3 2$$

$$h'(x) = \frac{1}{x \ln 3} + \frac{1}{2} \cdot \frac{1}{(x-1) \ln 3} - 0$$

$$= \frac{1}{\ln 3} \left[\frac{1}{x} + \frac{1}{2(x-1)} \right]$$

$$= \frac{1}{\ln 3} \left[\frac{3x-2}{2x(x-1)} \right]$$

$$45. y = \log_5 \sqrt{x^2-1} = \frac{1}{2} \log_5(x^2-1)$$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2-1) \ln 5} = \frac{x}{(x^2-1) \ln 5}$$

$$46. y = \log_{10} \frac{x^2-1}{x}$$

$$= \log_{10}(x^2-1) - \log_{10} x$$

$$\frac{dy}{dx} = \frac{2x}{(x^2-1) \ln 10} - \frac{1}{x \ln 10}$$

$$= \frac{1}{\ln 10} \left[\frac{2x}{x^2-1} - \frac{1}{x} \right]$$

$$= \frac{1}{\ln 10} \left[\frac{x^2+1}{x(x^2-1)} \right]$$

$$47. g(t) = \frac{10 \log_4 t}{t} = \frac{10}{\ln 4} \left(\frac{\ln t}{t} \right)$$

$$g'(t) = \frac{10}{\ln 4} \left[\frac{t(1/t) - \ln t}{t^2} \right]$$

$$= \frac{10}{t^2 \ln 4} [1 - \ln t]$$

$$= \frac{5}{t^2 \ln 2} (1 - \ln t)$$

$$48. f(t) = t^{3/2} \log_2 \sqrt{t+1} = t^{3/2} \frac{1}{2} \frac{\ln(t+1)}{\ln 2}$$

$$f'(t) = \frac{1}{2 \ln 2} \left[t^{3/2} \frac{1}{t+1} + \frac{3}{2} t^{1/2} \ln(t+1) \right]$$

49. $y = 2^{-x}, (-1, 2)$

$$y' = -2^{-x} \ln(2)$$

At $(-1, 2), y' = -2 \ln(2)$.

Tangent line: $y - 2 = -2 \ln(2)(x + 1)$

$$y = [-2 \ln(2)]x + 2 - 2 \ln(2)$$

50. $y = 5^{x-2}, (2, 1)$

$$y' = 5^{x-2} \ln 5$$

At $(2, 1), y' = \ln 5$.

Tangent line: $y - 1 = \ln(5)(x - 2)$

$$y = [\ln(5)]x + 1 - 2 \ln(5)$$

51. $y = \log_3 x, (27, 3)$

$$y' = \frac{1}{x \ln 3}$$

At $(27, 3), y' = \frac{1}{27 \ln 3}$.

Tangent line: $y - 3 = \frac{1}{27 \ln 3}(x - 27)$

$$y = \frac{1}{27 \ln 3}x + 3 - \frac{1}{\ln 3}$$

52. $y = \log_{10}(2x), (5, 1)$

$$y' = \frac{1}{x \ln 10}$$

At $(5, 1), y' = \frac{1}{5 \ln 10}$.

Tangent line: $y - 1 = \frac{1}{5 \ln 10}(x - 5)$

$$y = \frac{1}{5 \ln 10}x + 1 - \frac{1}{\ln 10}$$

53. $y = x^{2/x}$

$$\ln y = \frac{2}{x} \ln x$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} \left(\frac{1}{x} \right) + \ln x \left(-\frac{2}{x^2} \right) = \frac{2}{x^2} (1 - \ln x)$$

$$\frac{dy}{dx} = \frac{2y}{x^2} (1 - \ln x) = 2x^{(2/x)-2} (1 - \ln x)$$

54. $y = x^{x-1}$

$$\ln y = (x - 1)(\ln x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x - 1) \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = y \left[\frac{x - 1}{x} + \ln x \right]$$

$$= x^{x-2} (x - 1 + x \ln x)$$

55. $y = (x - 2)^{x+1}$

$$\ln y = (x + 1) \ln(x - 2)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x + 1) \left(\frac{1}{x - 2} \right) + \ln(x - 2)$$

$$\frac{dy}{dx} = y \left[\frac{x + 1}{x - 2} + \ln(x - 2) \right]$$

$$= (x - 2)^{x+1} \left[\frac{x + 1}{x - 2} + \ln(x - 2) \right]$$

56. $y = (1 + x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1 + x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} \left(\frac{1}{1 + x} \right) + \ln(1 + x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{y}{x} \left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x} \right]$$

$$= \frac{(1 + x)^{1/x}}{x} \left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x} \right]$$

57. $y = x^{\sin x}, \left(\frac{\pi}{2}, \frac{\pi}{2} \right)$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \frac{\sin x}{x} + \cos x \ln x$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2} \right): \frac{y'}{(\pi/2)} = \frac{1}{(\pi/2)} + 0$

$$y' = 1$$

Tangent line: $y - \frac{\pi}{2} = 1 \left(x - \frac{\pi}{2} \right)$

$$y = x$$

58. $y = (\sin x)^{2x}, \left(\frac{\pi}{2}, 1 \right)$

$$\ln y = 2x \ln(\sin x)$$

$$\frac{y'}{y} = \frac{2x}{\sin x} \cos x + 2 \ln(\sin x)$$

At $\left(\frac{\pi}{2}, 1 \right), y' = 0$.

Tangent line: $y = 1$

$$59. \quad y = (\ln x)^{\cos x}, \quad (e, 1)$$

$$\ln y = \cos x \cdot \ln(\ln x)$$

$$\frac{y'}{y} = \cos x \cdot \frac{1}{x \ln x} - \sin x \cdot \ln(\ln x)$$

$$\text{At } (e, 1), \quad y' = \cos(e) \frac{1}{e} - 0.$$

$$\text{Tangent line: } y - 1 = \frac{\cos(e)}{e}(x - e)$$

$$y = \frac{\cos(e)}{e}x + 1 - \cos(e)$$

$$60. \quad y = x^{1/x}, \quad (1, 1)$$

$$\ln y = \frac{1}{x} \ln x$$

$$\frac{y'}{y} = \frac{1}{x^2} - \frac{\ln x}{x^2}$$

$$\text{At } (1, 1), \quad y' = 1 - 0 = 1.$$

$$\text{Tangent line: } y - 1 = 1(x - 1)$$

$$y = x$$

$$61. \quad \int 3^x dx = \frac{3^x}{\ln 3} + C$$

$$62. \quad \int 5^{-x} dx = \frac{-5^{-x}}{\ln 5} + C$$

$$\begin{aligned} 63. \quad \int x(5^{-x^2}) dx &= -\frac{1}{2} \int 5^{-x^2}(-2x) dx \\ &= -\left(\frac{1}{2}\right) \frac{5^{-x^2}}{\ln 5} + C \\ &= \frac{-1}{2 \ln 5} (5^{-x^2}) + C \end{aligned}$$

$$\begin{aligned} 64. \quad \int (3-x) 7^{(3-x)^2} dx &= -\frac{1}{2} \int -2(3-x) 7^{(3-x)^2} dx \\ &= -\frac{1}{2 \ln 7} [7^{(3-x)^2}] + C \end{aligned}$$

$$\begin{aligned} 65. \quad \int \frac{3^{2x}}{1+3^{2x}} dx, \quad u &= 1+3^{2x}, \quad du = 2(\ln 3)3^{2x} dx \\ \frac{1}{2 \ln 3} \int \frac{(2 \ln 3)3^{2x}}{1+3^{2x}} dx &= \frac{1}{2 \ln 3} \ln(1+3^{2x}) + C \end{aligned}$$

$$\begin{aligned} 66. \quad \int 2^{\sin x} \cos x dx, \quad u &= \sin x, \quad du = \cos x dx \\ \frac{1}{\ln 2} 2^{\sin x} + C \end{aligned}$$

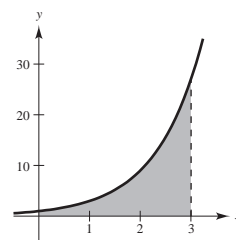
$$\begin{aligned} 67. \quad \int_{-1}^2 2^x dx &= \left[\frac{2^x}{\ln 2} \right]_{-1}^2 \\ &= \frac{1}{\ln 2} \left[4 - \frac{1}{2} \right] = \frac{7}{2 \ln 2} = \frac{7}{\ln 4} \end{aligned}$$

$$\begin{aligned} 68. \quad \int_{-2}^2 4^{x/2} dx &= 2 \int_{-2}^2 4^{x/2} \left(\frac{1}{2} dx \right) \\ &= \left[2 \frac{1}{\ln 4} 4^{x/2} \right]_{-2}^2 \\ &= \left[\frac{1}{\ln 2} 4^{x/2} \right]_{-2}^2 \\ &= \frac{1}{\ln 2} \left[4 - \frac{1}{4} \right] = \frac{15}{4 \ln 2} \end{aligned}$$

$$\begin{aligned} 69. \quad \int_0^1 (5^x - 3^x) dx &= \left[\frac{5^x}{\ln 5} - \frac{3^x}{\ln 3} \right]_0^1 \\ &= \left(\frac{5}{\ln 5} - \frac{3}{\ln 3} \right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 3} \right) \\ &= \frac{4}{\ln 5} - \frac{2}{\ln 3} \end{aligned}$$

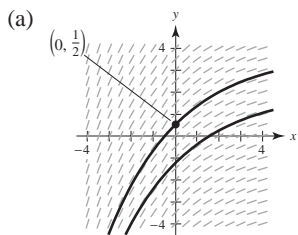
$$\begin{aligned} 70. \quad \int_1^e (6^x - 2^x) dx &= \left[\frac{6^x}{\ln 6} - \frac{2^x}{\ln 2} \right]_1^e \\ &= \left(\frac{6^e}{\ln 6} - \frac{2^e}{\ln 2} \right) - \left(\frac{6}{\ln 6} - \frac{2}{\ln 2} \right) \end{aligned}$$

$$\begin{aligned} 71. \quad \text{Area} &= \int_0^3 3^x dx \\ &= \left[\frac{1}{\ln 3} 3^x \right]_0^3 \\ &= \frac{1}{\ln 3} (27 - 1) = \frac{26}{\ln 3} \end{aligned}$$

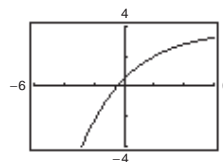


$$\begin{aligned}
 72. \text{ Area} &= \int_0^{\pi} 3^{\cos x} \sin x \, dx \\
 &= \left[\frac{-3^{\cos x}}{\ln 3} \right]_0^{\pi} \\
 &= \frac{-1}{\ln 3} [3^{-1} - 3] \\
 &= \frac{8}{3 \ln 3} \approx 2.4273
 \end{aligned}$$

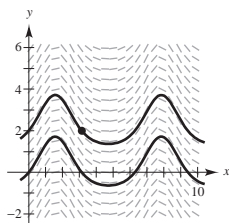
$$73. \frac{dy}{dx} = 0.4^{x/3}, \quad \left(0, \frac{1}{2}\right)$$



$$\begin{aligned}
 \text{(b)} \quad y &= \int 0.4^{x/3} \, dx = 3 \int 0.4^{x/3} \left(\frac{1}{3} \, dx\right) \\
 &= \frac{3}{\ln 0.4} 0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C \\
 \frac{1}{2} &= 3(\ln 2.5) + C \Rightarrow C = \frac{1}{2} - 3 \ln 2.5 \\
 y &= 3(\ln 2.5)(0.4)^{x/3} + \frac{1}{2} - 3 \ln 2.5 \\
 &= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}
 \end{aligned}$$



74. (a)



$$\text{(b)} \quad \frac{dy}{dx} = e^{\sin x} \cos x, \quad (\pi, 2)$$

$$y = \int e^{\sin x} \cos x \, dx = e^{\sin x} + C$$

$$(\pi, 2): 2 = e^{\sin \pi} + C = 1 + C \Rightarrow C = 1$$

$$y = e^{\sin x} + 1$$

$$76. f(x) = \log_{10} x$$

 (a) Domain: $x > 0$

$$\text{(b)} \quad y = \log_{10} x$$

$$10^y = x$$

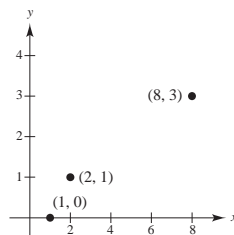
$$f^{-1}(x) = 10^x$$

$$\text{(c)} \quad \log_{10} 1000 = \log_{10} 10^3 = 3$$

$$\log_{10} 10,000 = \log_{10} 10^4 = 4$$

If $1000 \leq x \leq 10,000$, then $3 \leq f(x) \leq 4$.

75.



x	1	2	8
y	0	1	3

 (a) y is an exponential function of x : False

 (b) y is a logarithmic function of x : True; $y = \log_2 x$

 (c) x is an exponential function of y : True, $2^y = x$

 (d) y is a linear function of x : False

 (d) If $f(x) < 0$, then $0 < x < 1$.

$$\text{(e)} \quad f(x) + 1 = \log_{10} x + \log_{10} 10$$

$$= \log_{10}(10x)$$

x must have been increased by a factor of 10.

$$\text{(f)} \quad \log_{10} \left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$$

$$= 3n - n = 2n$$

Thus, $x_1/x_2 = 10^{2n} = 100^n$.

$$77. f(x) = \log_2 x \Rightarrow f'(x) = \frac{1}{x \ln 2}$$

$$g(x) = x^x \Rightarrow g'(x) = x^x(1 + \ln x)$$

Note: Let $y = g(x)$. Then:

$$\ln y = \ln x^x = x \ln x$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x) = g'(x)$$

$$h(x) = x^2 \Rightarrow h'(x) = 2x$$

$$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$$

From greatest to smallest rate of growth:

$$g(x), k(x), h(x), f(x)$$

$$78. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a)a^x$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$(d) y = a^a$$

$$y' = 0$$

$$79. C(t) = P(1.05)^t$$

$$(a) C(10) = 24.95(1.05)^{10} \\ \approx \$40.64$$

$$(b) \frac{dC}{dt} = P(\ln 1.05)(1.05)^t$$

$$\text{When } t = 1, \frac{dC}{dt} \approx 0.051P.$$

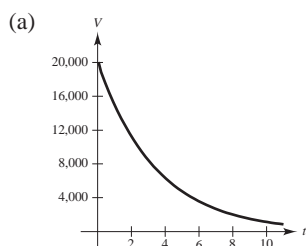
$$\text{When } t = 8, \frac{dC}{dt} \approx 0.072P.$$

$$(c) \frac{dC}{dt} = (\ln 1.05)[P(1.05)^t]$$

$$= (\ln 1.05)C(t)$$

The constant of proportionality is $\ln 1.05$.

$$80. V(t) = 20,000\left(\frac{3}{4}\right)^t$$

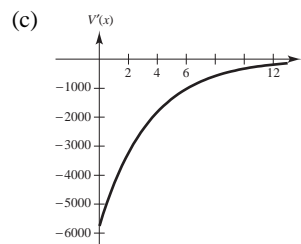


$$V(2) = 20,000\left(\frac{3}{4}\right)^2 = \$11,250$$

$$(b) \frac{dV}{dt} = 20,000\left(\ln \frac{3}{4}\right)\left(\frac{3}{4}\right)^t$$

$$\text{When } t = 1, \frac{dV}{dt} \approx -4315.23.$$

$$\text{When } t = 4, \frac{dV}{dt} \approx -1820.49.$$



Horizontal asymptote: $V' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

$$81. P = \$1000, r = 3\frac{1}{2}\% = 0.035, t = 10$$

$$A = 1000\left(1 + \frac{0.035}{n}\right)^{10n}$$

$$A = 1000e^{(0.035)(10)} = 1419.07$$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

$$82. P = \$2500, r = 6\% = 0.06, t = 20$$

$$A = 2500\left(1 + \frac{0.06}{n}\right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

83. $P = \$1000, r = 5\% = 0.05, t = 30$

$$A = 1000 \left(1 + \frac{0.05}{n} \right)^{30n}$$

$$A = 1000e^{(0.05)30} = 4481.69$$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

84. $P = \$5000, r = 7\% = 0.07, t = 25$

$$A = 5000 \left(1 + \frac{0.07}{n} \right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

85. $100,000 = Pe^{0.05t} \Rightarrow P = 100,000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,533.53	8208.50

86. $100,000 = Pe^{0.06t} \Rightarrow P = 100,000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

87. $100,000 = P \left(1 + \frac{0.05}{12} \right)^{12t} \Rightarrow P = 100,000 \left(1 + \frac{0.05}{12} \right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

88. $100,000 = P \left(1 + \frac{0.07}{365} \right)^{365t} \Rightarrow P = 100,000 \left(1 + \frac{0.07}{365} \right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

89. (a) $A = 20,000 \left(1 + \frac{0.06}{365} \right)^{(365)(8)} \approx \$32,320.21$

(b) $A = \$30,000$

(c) $A = 8000 \left(1 + \frac{0.06}{365} \right)^{(365)(8)} + 20,000 \left(1 + \frac{0.06}{365} \right)^{(365)(4)}$
 $\approx \$12,928.09 + 25,424.48 = \$38,352.57$

(d) $A = 9000 \left[\left(1 + \frac{0.06}{365} \right)^{(365)(8)} + \left(1 + \frac{0.06}{365} \right)^{(365)(4)} + 1 \right]$
 $\approx \$34,985.11$

Take option (c).

90. Let $P = \$100, 0 \leq t \leq 20$.

(a) $A = 100e^{0.03t}$

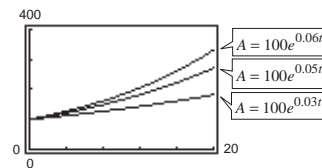
$$A(20) \approx 182.21$$

(b) $A = 100e^{0.05t}$

$$A(20) \approx 271.83$$

(c) $A = 100e^{0.06t}$

$$A(20) \approx 332.01$$



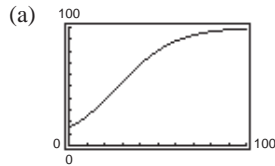
91. (a) $\lim_{t \rightarrow \infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \frac{322.27}{t^2} e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

93. $y = \frac{300}{3 + 17e^{-0.0625x}}$



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$

$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$

$17e^{-0.0625x} - 3 = 0 \Rightarrow$

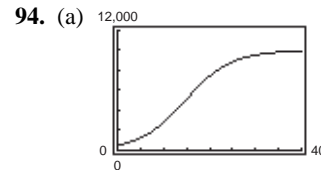
$x \approx 27.8$ or 27,800 egg masses.

92. (a) $\lim_{n \rightarrow \infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})^2} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})^2}$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$



(b) Limiting size: 10,000 fish

(c) $p(t) = \frac{10,000}{1 + 19e^{-t/5}}$

$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2} \left(\frac{19}{5} \right) (10,000) = \frac{38,000e^{-t/5}}{(1 + 19e^{-t/5})^2}$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

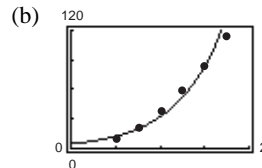
(d) $p''(t) = -\frac{38,000}{5}(e^{-t/5}) \left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3} \right] = 0$

$19e^{-t/5} = 1$

$\frac{t}{5} = \ln 19$

$t = 5 \ln 19 \approx 14.72$

95. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$



(c) $B'(d) = 9.0952e^{1.9132d}$

$B'(0.8) \approx 42.03$ tons/inch

$B'(1.5) \approx 160.38$ tons/inch

96. (a) $y_1 = 16.32t + 43.4$, linear

$y_2 = -93.58 + 131.22 \ln x$

$y_3 = (80.99)1.097^x$

$y_4 = (36.55)x^{0.754}$

(c) The amount given increases 16.32 billion on average per year.

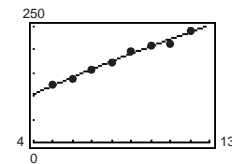
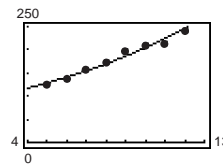
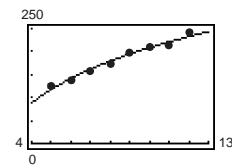
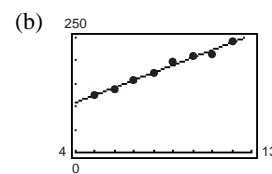
(d) $y_1'(6) = 16.32$

$y_2'(6) = \frac{131.22}{6} = 21.87$

$y_3'(6) = (80.99)1.097^6(\ln 1.097) \approx 13.07$

$y_4'(6) = 36.55(0.754)6^{0.754-1} \approx 17.74$

y_2 is increasing at the greatest rate.

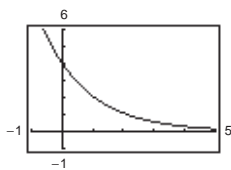


97. (a) $\int_0^4 f(t) dt \approx 5.67$

$$\int_0^4 g(t) dt \approx 5.67$$

$$\int_0^4 h(t) dt \approx 5.67$$

(b)


 (c) The functions appear to be equal: $f(t) = g(t) = h(t)$. Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t \approx 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t \approx 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

 98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1+x)^{1/x}$	2	2.594	2.705	2.718	2.718

 99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$$y = C(k^t)$$

$$\text{When } t = 0, y = 1200 \Rightarrow C = 1200.$$

$$y = 1200(k^t)$$

$$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$$

$$\text{Let } k = 0.6.$$

$$y = 1200(0.6)^t$$

 100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$$y = C(k^t)$$

$$\text{When } t = 0, y = 600 \Rightarrow C = 600.$$

$$y = 600(k^t)$$

$$\frac{630}{600} = 1.05, \frac{661.50}{630} = 1.05, \frac{694.58}{661.50} \approx 1.05,$$

$$\frac{729.30}{694.58} \approx 1.05$$

$$\text{Let } k = 1.05.$$

$$y = 600(1.05)^t$$

 101. False. e is an irrational number.

102. True

$$\begin{aligned} f(e^{n+1}) - f(e^n) &= \ln e^{n+1} - \ln e^n \\ &= n + 1 - n \\ &= 1 \end{aligned}$$

103. True

$$\begin{aligned} f(g(x)) &= 2 + e^{\ln(x-2)} \\ &= 2 + x - 2 = x \\ g(f(x)) &= \ln(2 + e^x - 2) \\ &= \ln e^x = x \end{aligned}$$

104. True

$$\frac{d^n y}{dx^n} = Ce^x$$

$$= y \text{ for } n = 1, 2, 3, \dots$$

105. True

$$\frac{d}{dx}[e^x] = e^x \text{ and } \frac{d}{dx}[e^{-x}] = -e^{-x}$$

$$e^x = e^{-x} \text{ when } x = 0.$$

$$(e^0)(-e^{-0}) = -1$$

106. True

$$f(x) = g(x)e^x = 0 \Rightarrow$$

$$g(x) = 0 \text{ since } e^x > 0 \text{ for all } x.$$

107. $\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1$

$$\frac{dy}{y\left[\frac{5}{4} - y\right]} = \frac{8}{25} dt \Rightarrow \frac{4}{5} \int \left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right) dy = \int \frac{8}{25} dt \Rightarrow$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{\frac{5}{4} - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{\frac{5}{4} - y} = e^{(2/5)t+C} = C_1 e^{(2/5)t}$$

$$y(0) = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{\frac{5}{4} - y}$$

$$\Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\Rightarrow y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

108. $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

109. (a) $y^x = x^y$

$$x \ln y = y \ln x$$

$$x \frac{y'}{y} + \ln y = \frac{y}{x} + y' \ln x$$

$$y' \left[\frac{x}{y} - \ln x \right] = \frac{y}{x} - \ln y$$

$$y' = \frac{(y/x) - \ln y}{(x/y) - \ln x}$$

$$y' = \frac{y^2 - xy \ln y}{x^2 - xy \ln x}$$

(b) (i) At (c, c) : $y' = \frac{c^2 - c^2 \ln c}{c^2 - c^2 \ln c} = 1, (c \neq 0, e)$

(ii) At $(2, 4)$: $y' = \frac{16 - 8 \ln 4}{4 - 8 \ln 2} = \frac{4 - 4 \ln 2}{1 - 2 \ln 2} \approx -3.1774$

(iii) At $(4, 2)$: $y' = \frac{4 - 8 \ln 2}{16 - 8 \ln 4} = \frac{1 - 2 \ln 2}{4 - 4 \ln 2} \approx -0.3147$

(c) y' is undefined for

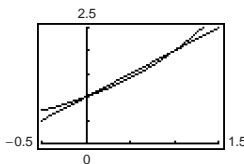
$$x^2 = xy \ln x \Rightarrow x = y \ln x = \ln x^y \Rightarrow e^x = x^y.$$

At (e, e) , y' is undefined.

110. $f(x) = 1 + x, g(x) = b^x$

(a) $b = 2$

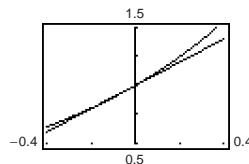
$$g(x) = 2^x$$



Intersection points: $(0, 1), (1, 2)$

(b) $b = 3$

$$g(x) = 3^x$$



Intersection point: $(0, 1)$

(c) $g(x) = e^x \geq 1 + x$ for all x .

$$g'(0) = 1 = f'(0)$$

Hence, $g(x) \geq f(x)$ for all $b \geq e$.

111. Let $f(x) = \frac{\ln x}{x}$, $x > 0$.

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \text{ for } x > e \Rightarrow f \text{ is decreasing for}$$

$x \geq e$. Hence, for $e \leq x < y$:

$$f(x) > f(y)$$

$$\frac{\ln x}{x} > \frac{\ln y}{y}$$

$$(xy) \frac{\ln x}{x} > (xy) \frac{\ln y}{y}$$

$$\ln x^y > \ln y^x$$

$$x^y > y^x$$

For $n \geq 8$, $e < \sqrt{n} < \sqrt{n+1}$, ($\sqrt{8} \approx 2.828$) and so letting $x = \sqrt{n}$, $y = \sqrt{n+1}$, we have

$$(\sqrt{n})^{\sqrt{n+1}} > (\sqrt{n+1})^{\sqrt{n}}$$

Note: $\sqrt{8}^{\sqrt{9}} \approx 22.6$ and $\sqrt{9}^{\sqrt{8}} \approx 22.4$.

Note: This same argument shows $e^\pi > \pi^e$.

112. $\log_e \left(1 + \frac{1}{x}\right) = \ln \left(1 + \frac{1}{x}\right)$

$$= \int_x^{1+x} \frac{dt}{t}$$

$$> \int_x^{1+x} \frac{dt}{1+x} \quad \left(\text{because } 1+x \geq t \text{ on } x \leq t \leq 1+x\right)$$

$$= \left[\frac{t}{1+x} \right]_x^{1+x}$$

$$= \frac{1+x}{1+x} - \frac{x}{1+x}$$

$$= \frac{1}{1+x}$$

Note: You can confirm this result by graphing

$$y_1 = \ln \left(1 + \frac{1}{x}\right) \text{ and}$$

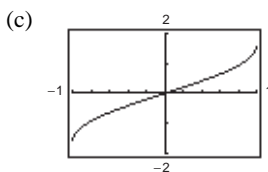
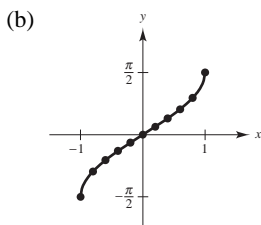
$$y_2 = \frac{1}{1+x}.$$

Section 5.6 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571



(d) Symmetric about origin:

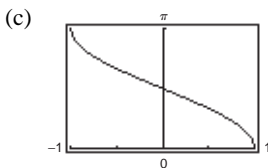
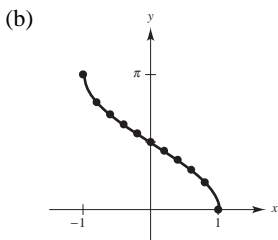
$$\arcsin(-x) = -\arcsin x$$

Intercept: (0, 0)

2. $y = \arccos x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.644	0



(d) Intercepts: $\left(0, \frac{\pi}{2}\right)$ and (1, 0)

No symmetry

3. $y = \arccos x$

$$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right) \text{ because } \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\left(\frac{1}{2}, \frac{\pi}{3}\right) \text{ because } \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right) \text{ because } \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

4. $\left(_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$$\left(_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$$

$$\left(-\sqrt{3}, _\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$$

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

6. $\arcsin 0 = 0$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

8. $\arccos 0 = \frac{\pi}{2}$

9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

10. $\operatorname{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

12. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

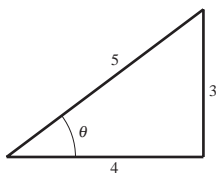
13. $\arccos(-0.8) \approx 2.50$

14. $\arcsin(-0.39) \approx -0.40$

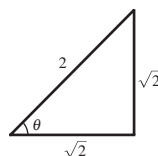
15. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

16. $\arctan(-3) \approx -1.25$

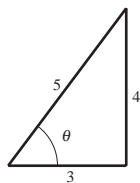
17. (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$



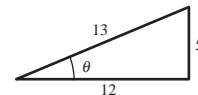
18. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$



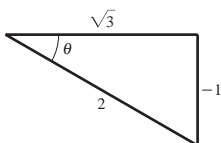
(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$



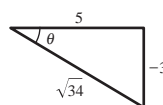
(b) $\cos\left(\arcsin \frac{5}{13}\right) = \frac{12}{13}$



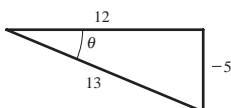
19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



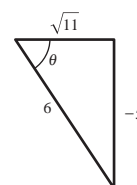
20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \frac{\sqrt{34}}{5}$



(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



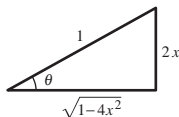
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right] = -\frac{5\sqrt{11}}{11}$



21. $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

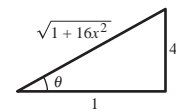
$y = \cos \theta = \sqrt{1 - 4x^2}$



22. $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

$y = \sec \theta = \sqrt{1 + 16x^2}$

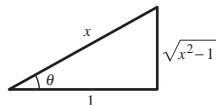


23. $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x, 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$

$y = \sin \theta = \frac{\sqrt{x^2 - 1}}{|x|}$

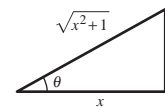
The absolute value bars on x are necessary because of the restriction $0 \leq \theta \leq \pi, \theta \neq \pi/2$, and $\sin \theta$ for this domain must always be nonnegative.



24. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

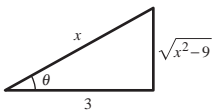
$y = \cos \theta = \frac{x}{\sqrt{x^2 + 1}}$



25. $y = \tan\left(\operatorname{arcsec} \frac{x}{3}\right)$

$\theta = \operatorname{arcsec} \frac{x}{3}$

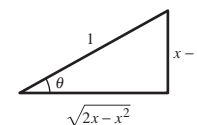
$y = \tan \theta = \frac{\sqrt{x^2 - 9}}{3}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

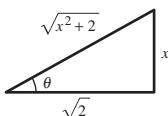
$y = \sec \theta = \frac{1}{\sqrt{2x - x^2}}$



27. $y = \csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

$\theta = \arctan \frac{x}{\sqrt{2}}$

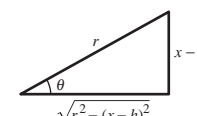
$y = \csc \theta = \frac{\sqrt{x^2 + 2}}{x}$



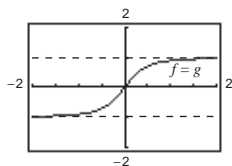
28. $y = \cos\left(\arcsin \frac{x - h}{r}\right)$

$\theta = \arcsin \frac{x - h}{r}$

$y = \cos \theta = \frac{\sqrt{r^2 - (x - h)^2}}{r}$



29. (a)



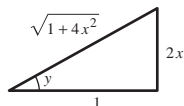
(b) Let $y = \arctan(2x)$

$\tan y = 2x$

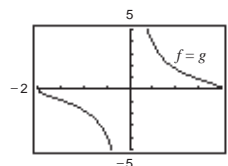
$\sin y = \sin(\arctan(2x))$

$= \frac{2x}{\sqrt{1 + 4x^2}}$

(c) Asymptotes: $y = \pm 1$



30. (a)

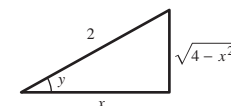


(b) Let $y = \arccos \frac{x}{2}$

$\cos y = \frac{x}{2}$

$\tan y = \tan\left(\arccos \frac{x}{2}\right)$

$= \frac{\sqrt{4 - x^2}}{x}$



(c) No horizontal asymptotes; domain is $-2 \leq x \leq 0$, $0 < x \leq 2$.

(Vertical asymptote: $x = 0$)

$$31. \arcsin(3x - \pi) = \frac{1}{2}$$

$$3x - \pi = \sin\left(\frac{1}{2}\right)$$

$$x = \frac{1}{3}[\sin\left(\frac{1}{2}\right) + \pi] \approx 1.207$$

$$33. \arcsin\sqrt{2x} = \arccos\sqrt{x}$$

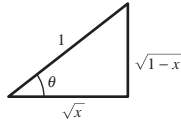
$$\sqrt{2x} = \sin(\arccos\sqrt{x})$$

$$\sqrt{2x} = \sqrt{1-x}, \quad 0 \leq x \leq 1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$



$$32. \arctan(2x - 5) = -1$$

$$2x - 5 = \tan(-1)$$

$$x = \frac{1}{2}(\tan(-1) + 5) \approx 1.721$$

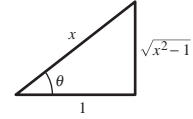
$$34. \arccos x = \operatorname{arcsec} x$$

$$x = \cos(\operatorname{arcsec} x)$$

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$



$$35. (a) \operatorname{arccsc} x = \arcsin \frac{1}{x}, \quad |x| \geq 1$$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \leq y < 0 \text{ and } 0 < y \leq \frac{\pi}{2},$$

$$\csc y = x \implies \sin y = 1/x. \text{ Thus, } y = \arcsin(1/x).$$

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

$$(b) \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x) \tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

$$36. (a) \arcsin(-x) = -\arcsin x, \quad |x| \leq 1$$

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \implies x = -\sin y \implies x = \sin(-y).$$

Thus, $-y = \arcsin x \implies y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

$$(b) \arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$$

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \implies x = -\cos y \implies x = \cos(\pi - y).$$

Thus, $\pi - y = \arccos x \implies y = \pi - \arccos x$.

Therefore, $\arccos(-x) = \pi - \arccos x$.

$$37. f(x) = \arcsin(x - 1)$$

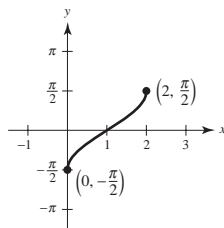
$$x - 1 = \sin y$$

$$x = 1 + \sin y$$

$$\text{Domain: } [0, 2]$$

$$\text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.



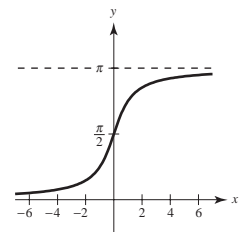
$$38. f(x) = \arctan x + \frac{\pi}{2}$$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

$$\text{Domain: } (-\infty, \infty)$$

$$\text{Range: } (0, \pi)$$

$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.



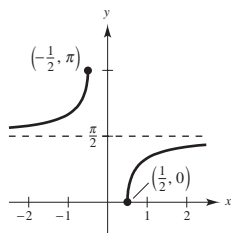
39. $f(x) = \operatorname{arcsec} 2x$

$$2x = \sec y$$

$$x = \frac{1}{2} \sec y$$

Domain: $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

Range: $\left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$



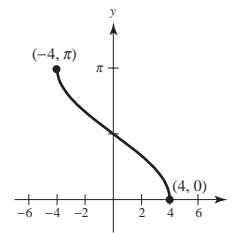
40. $f(x) = \arccos\left(\frac{x}{4}\right)$

$$\frac{x}{4} = \cos y$$

$$x = 4 \cos y$$

Domain: $[-4, 4]$

Range: $[0, \pi]$



41. $f(x) = 2 \arcsin(x - 1)$

$$f'(x) = \frac{2}{\sqrt{1 - (x - 1)^2}}$$

$$= \frac{2}{\sqrt{2x - x^2}}$$

42. $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1 - t^4}}$$

43. $g(x) = 3 \arccos \frac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4 - x^2}}$$

44. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x| \sqrt{4x^2 - 1}}$$

$$= \frac{1}{|x| \sqrt{4x^2 - 1}}$$

45. $f(x) = \arctan \frac{x}{a}$

$$f'(x) = \frac{1/a}{1 + (x^2/a^2)} = \frac{a}{a^2 + x^2}$$

46. $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1+x}\right) \left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}(1+x)}$$

47. $g(x) = \frac{\arcsin 3x}{x}$

$$g'(x) = \frac{x(3/\sqrt{1-9x^2}) - \arcsin 3x}{x^2}$$

$$= \frac{3x - \sqrt{1-9x^2} \arcsin 3x}{x^2 \sqrt{1-9x^2}}$$

48. $h(x) = x^2 \arctan x$

$$h'(x) = 2x \arctan x + \frac{x^2}{1+x^2}$$

49. $h(t) = \sin(\arccos t) = \sqrt{1-t^2}$

$$h'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t)$$

$$= \frac{-t}{\sqrt{1-t^2}}$$

50. $f(x) = \arcsin x + \arccos x = \frac{\pi}{2}$

$$f'(x) = 0$$

51. $y = x \arccos x - \sqrt{1-x^2}$

$$y' = \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \arccos x$$

52. $y = \ln(t^2 + 4) - \frac{1}{2} \arctan \frac{t}{2}$

$$y' = \frac{2t}{t^2 + 4} - \frac{1}{2} \cdot \frac{1}{1 + (t/2)^2} \left(\frac{1}{2}\right)$$

$$= \frac{2t}{t^2 + 4} - \frac{1}{t^2 + 4} = \frac{2t - 1}{t^2 + 4}$$

53. $y = \frac{1}{2} \left(\frac{1}{2} \ln \frac{x+1}{x-1} + \arctan x \right)$

$$= \frac{1}{4} [\ln(x+1) - \ln(x-1)] + \frac{1}{2} \arctan x$$

$$\frac{dy}{dx} = \frac{1}{4} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

$$54. y = \frac{1}{2} \left[x\sqrt{4-x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]$$

$$\begin{aligned} y' &= \frac{1}{2} \left[x \frac{1}{2} (4-x^2)^{-1/2} (-2x) + \sqrt{4-x^2} + 2 \frac{1}{\sqrt{1-(x/2)^2}} \right] \\ &= \frac{1}{2} \left[\frac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} + \frac{4}{\sqrt{4-x^2}} \right] \\ &= \sqrt{4-x^2} \end{aligned}$$

$$55. y = x \arcsin x + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = x \left(\frac{1}{\sqrt{1-x^2}} \right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

$$56. y = x \arctan 2x - \frac{1}{4} \ln(1+4x^2)$$

$$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4} \left(\frac{8x}{1+4x^2} \right) = \arctan(2x)$$

$$57. y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$$

$$\begin{aligned} y' &= 2 \frac{1}{\sqrt{1-(x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4} (16-x^2)^{-1/2} (-2x) \\ &= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} \\ &= \frac{16 - (16-x^2) + x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}} \end{aligned}$$

$$58. y = 25 \arcsin \frac{x}{5} - x\sqrt{25-x^2}$$

$$\begin{aligned} y' &= 5 \frac{1}{\sqrt{1-(x/5)^2}} - \sqrt{25-x^2} - x \frac{1}{2} (25-x^2)^{-1/2} (-2x) \\ &= \frac{25}{\sqrt{25-x^2}} - \frac{(25-x^2)}{\sqrt{25-x^2}} + \frac{x^2}{\sqrt{25-x^2}} \\ &= \frac{2x^2}{\sqrt{25-x^2}} \end{aligned}$$

$$59. y = \arctan x + \frac{x}{1+x^2}$$

$$\begin{aligned} y' &= \frac{1}{1+x^2} + \frac{(1+x^2) - x(2x)}{(1+x^2)^2} \\ &= \frac{(1+x^2) + (1-x^2)}{(1+x^2)^2} \\ &= \frac{2}{(1+x^2)^2} \end{aligned}$$

$$60. y = \arctan \frac{x}{2} - \frac{1}{2(x^2+4)}$$

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{1+(x/2)^2} + \frac{1}{2} (x^2+4)^{-2} (2x) \\ &= \frac{2}{x^2+4} + \frac{x}{(x^2+4)^2} \\ &= \frac{2x^2+8+x}{(x^2+4)^2} \end{aligned}$$

$$61. y = 2 \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{3} \right)$$

$$y' = \frac{2}{\sqrt{1-x^2}}$$

$$\text{At } \left(\frac{1}{2}, \frac{\pi}{3} \right), y' = \frac{2}{\sqrt{1-(1/4)}} = \frac{4}{\sqrt{3}}$$

$$\text{Tangent line: } y - \frac{\pi}{3} = \frac{4}{\sqrt{3}} \left(x - \frac{1}{2} \right)$$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

$$62. y = \frac{1}{2} \arccos x, \quad \left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8} \right)$$

$$y' = \frac{-1}{2\sqrt{1-x^2}}$$

$$\text{At } \left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8} \right), y' = \frac{-1}{2\sqrt{1/2}} = -\frac{\sqrt{2}}{2}$$

$$\text{Tangent line: } y - \frac{3\pi}{8} = -\frac{\sqrt{2}}{2} \left(x + \frac{\sqrt{2}}{2} \right)$$

$$y = -\frac{\sqrt{2}}{2}x + \frac{3\pi}{8} - \frac{1}{2}$$

$$63. y = \arctan\left(\frac{x}{2}\right), \quad \left(2, \frac{\pi}{4}\right)$$

$$y' = \frac{1}{1 + (x^2/4)} \left(\frac{1}{2}\right) = \frac{2}{4 + x^2}$$

$$\text{At } \left(2, \frac{\pi}{4}\right), y' = \frac{2}{4 + 4} = \frac{1}{4}$$

$$\text{Tangent line: } y - \frac{\pi}{4} = \frac{1}{4}(x - 2)$$

$$y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$$

$$65. y = 4x \arccos(x - 1), \quad (1, 2\pi)$$

$$y' = 4x \frac{-1}{\sqrt{1 - (x - 1)^2}} + 4 \arccos(x - 1)$$

$$\text{At } (1, 2\pi), y' = -4 + 2\pi$$

$$\text{Tangent line: } y - 2\pi = (2\pi - 4)(x - 1)$$

$$y = (2\pi - 4)x + 4$$

$$67. f(x) = \arctan x, \quad a = 0$$

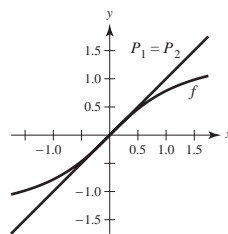
$$f(0) = 0$$

$$f'(x) = \frac{1}{1 + x^2}, \quad f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}, \quad f''(0) = 0$$

$$P_1(x) = f(0) + f'(0)x = x$$

$$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$$



$$68. f(x) = \arccos x, \quad a = 0$$

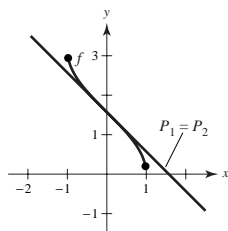
$$f(0) = \frac{\pi}{2}$$

$$f'(x) = \frac{-1}{\sqrt{1 - x^2}}, \quad f'(0) = -1$$

$$f''(x) = \frac{-x}{(1 - x^2)^{3/2}}, \quad f''(0) = 0$$

$$P_1(x) = f(0) + f'(0)x = \frac{\pi}{2} - x$$

$$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = \frac{\pi}{2} - x$$



$$64. y = \operatorname{arcsec}(4x), \quad \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$$

$$y' = \frac{4}{|4x|\sqrt{16x^2 - 1}} = \frac{1}{x\sqrt{16x^2 - 1}} \text{ for } x > 0$$

$$\text{At } \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right), y' = \frac{1}{(\sqrt{2}/4)\sqrt{2 - 1}} = 2\sqrt{2}$$

$$\text{Tangent line: } y - \frac{\pi}{4} = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{4}\right)$$

$$y = 2\sqrt{2}x + \frac{\pi}{4} - 1$$

$$66. y = 3x \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{4}\right)$$

$$y' = 3x \frac{1}{\sqrt{1 - x^2}} + 3 \arcsin x$$

$$\text{At } \left(\frac{1}{2}, \frac{\pi}{4}\right), y' = \frac{3}{2} \frac{1}{\sqrt{3/4}} + 3\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{2}$$

$$\text{Tangent line: } y - \frac{\pi}{4} = \left(\sqrt{3} + \frac{\pi}{2}\right)\left(x - \frac{1}{2}\right)$$

$$y = \left(\sqrt{3} + \frac{\pi}{2}\right)x - \frac{\sqrt{3}}{2}$$

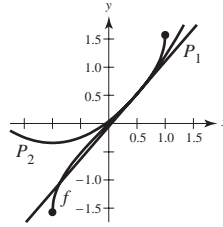
69. $f(x) = \arcsin x, \quad a = \frac{1}{2}$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$



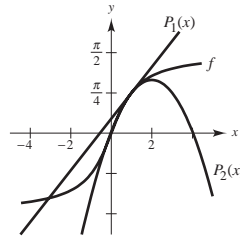
70. $f(x) = \arctan x, \quad a = 1$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x-1) = \frac{\pi}{4} + \frac{1}{2}(x-1)$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$$



71. $f(x) = \operatorname{arcsec} x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}} - 1$$

$$= 0 \text{ when } |x|\sqrt{x^2-1} = 1$$

$$x^2(x^2-1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm \sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: (1.272, -0.606)

Relative minimum: (-1.272, 3.747)

72. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - 2$$

$$= 0 \text{ when } \sqrt{1-x^2} = \frac{1}{2} \text{ or } x = \pm \frac{\sqrt{3}}{2}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

73. $f(x) = \arctan x - \arctan(x-4)$

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+(x-4)^2} = 0$$

$$1+x^2 = 1+(x-4)^2$$

$$0 = -8x + 16$$

$$x = 2$$

By the First Derivative Test, (2, 2.214) is a relative maximum.

74. $f(x) = \arcsin x - 2 \arctan x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2}{1+x^2} = 0$$

$$1+x^2 = 2\sqrt{1-x^2}$$

$$1+2x^2+x^4 = 4(1-x^2)$$

$$x^4+6x^2-3=0$$

$$x = \pm 0.681$$

By the First Derivative Test, (-0.681, 0.447) is a relative maximum and (0.681, -0.447) is a relative minimum.

$$75. \quad x^2 + x \arctan y = y - 1, \quad \left(-\frac{\pi}{4}, 1\right)$$

$$2x + \arctan y + \frac{x}{1+y^2}y' = y'$$

$$\left(1 - \frac{x}{1+y^2}\right)y' = 2x + \arctan y$$

$$y' = \frac{2x + \arctan y}{1 - \frac{x}{1+y^2}}$$

$$\text{At } \left(-\frac{\pi}{4}, 1\right), y' = \frac{-\frac{\pi}{2} + \frac{\pi}{4}}{1 - \frac{-\pi/4}{2}} = \frac{-\frac{\pi}{2}}{2 + \frac{\pi}{4}} = \frac{-2\pi}{8 + \pi}.$$

$$\text{Tangent line: } y - 1 = \frac{-2\pi}{8 + \pi}\left(x + \frac{\pi}{4}\right)$$

$$y = \frac{-2\pi}{8 + \pi}x + 1 - \frac{\pi^2}{16 + 2\pi}$$

$$77. \quad \arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}y' = 0$$

$$\frac{1}{\sqrt{1-y^2}}y' = \frac{-1}{\sqrt{1-x^2}}$$

$$\text{At } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad y' = -1$$

$$\text{Tangent line: } y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$

$$y = -x + \sqrt{2}$$

79. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

$$81. \quad y = \operatorname{arccot} x, \quad 0 < y < \pi$$

$$x = \cot y$$

$$\tan y = \frac{1}{x}$$

So, graph the function $y = \arctan(1/x)$ for $x > 0$ and $y = \arctan(1/x) + \pi$ for $x < 0$.

83. False

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

84. False

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \text{ so}$$

$$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

85. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} > 0 \text{ for all } x.$$

$$76. \quad \arctan(xy) = \arcsin(x+y), \quad (0, 0)$$

$$\frac{1}{1+(xy)^2}[y + xy'] = \frac{1}{\sqrt{1-(x+y)^2}}[1+y']$$

$$\text{At } (0, 0): 0 = 1 + y' \Rightarrow y' = -1$$

$$\text{Tangent line: } y = -x$$

$$78. \quad \arctan(x+y) = y^2 + \frac{\pi}{4}, \quad (1, 0)$$

$$\frac{1}{1+(x+y)^2}[1+y'] = 2yy'$$

$$\text{At } (1, 0): \frac{1}{2}[1+y'] = 0 \Rightarrow y' = -1$$

$$\text{Tangent line: } y - 0 = -1(x - 1)$$

$$y = -x + 1$$

80. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

82. The derivatives are algebraic. See Theorem 5.18.

86. False

 The range of $y = \arcsin x$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

87. True

$$\begin{aligned} \frac{d}{dx}[\arctan(\tan x)] &= \frac{\sec^2 x}{1 + \tan^2 x} \\ &= \frac{\sec^2 x}{\sec^2 x} = 1 \end{aligned}$$

88. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

89. (a) $\cot \theta = \frac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-1/5}{1 + (x/5)^2} \frac{dx}{dt} = \frac{-5}{x^2 + 25} \frac{dx}{dt}$

If $\frac{dx}{dt} = -400$ and $x = 10$, $\frac{d\theta}{dt} = 16$ rad/hr.

If $\frac{dx}{dt} = -400$ and $x = 3$, $\frac{d\theta}{dt} \approx 58.824$ rad/hr.

90. (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9} \frac{dx}{dt}$

If $x = 10$, $\frac{d\theta}{dt} \approx 11.001$ rad/hr.

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667$ rad/hr.

 A lower altitude results in a greater rate of change of θ .

91. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec

(b) $\tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$

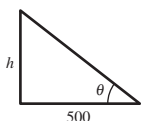
$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2}$$

$$= \frac{-1000t}{15,625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

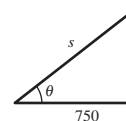
When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.



92. $\cos \theta = \frac{750}{s}$

$$\theta = \arccos\left(\frac{750}{s}\right)$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}} \left(\frac{-750}{s^2}\right) \frac{ds}{dt} \\ &= \frac{750}{s\sqrt{s^2 - 750^2}} \frac{ds}{dt} \end{aligned}$$



93. (a) $\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)}$

$$= \frac{x + y}{1 - xy}, \quad xy \neq 1$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), \quad xy \neq 1.$$

(b) Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2) + (1/3)}{1 - [(1/2) \cdot (1/3)]}$$

$$= \arctan\frac{5/6}{1 - (1/6)}$$

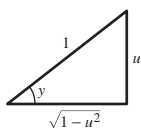
$$= \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

 94. (a) Let $y = \arcsin u$. Then

$\sin y = u$

$\cos y \cdot y' = u'$

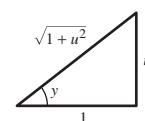
$$\frac{dy}{dx} = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1 - u^2}}$$


 (b) Let $y = \arctan u$. Then

$\tan y = u$

$\sec^2 y \frac{dy}{dx} = u'$

$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}$$



—CONTINUED—

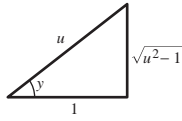
94. —CONTINUED—

 (c) Let $y = \operatorname{arcsec} u$. Then

$$\sec y = u$$

$$\sec y \tan y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}$$



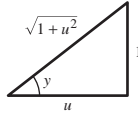
Note: The absolute value sign in the formula for the derivative of $\operatorname{arcsec} u$ is necessary because the inverse secant function has a positive slope at every value in its domain.

 (e) Let $y = \operatorname{arccot} u$. Then

$$\cot y = u$$

$$-\csc^2 y \frac{dy}{dx} = u'$$

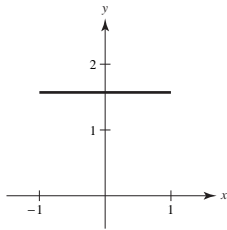
$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}$$


 95. $f(x) = kx + \sin x$

$$f'(x) = k + \cos x \geq 0 \text{ for } k \geq 1$$

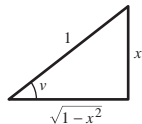
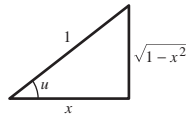
$$f'(x) = k + \cos x \leq 0 \text{ for } k \leq -1$$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

 97. (a) $f(x) = \arccos x + \arcsin x$

 (b) The graph of f is the constant function $y = \pi/2$.

 (c) Let $u = \arccos x$ and $v = \arcsin x$

$$\cos u = x \quad \text{and} \quad \sin v = x.$$



$$\begin{aligned} \sin(u+v) &= \sin u \cos v + \sin v \cos u \\ &= \sqrt{1-x^2} \sqrt{1-x^2} + x \cdot x \\ &= 1-x^2+x^2=1 \end{aligned}$$

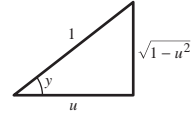
Hence, $u+v = \pi/2$. Thus, $\arccos x + \arcsin x = \pi/2$.

 (d) Let $y = \arccos u$. Then

$$\cos y = u$$

$$-\sin y \frac{dy}{dx} = u'$$

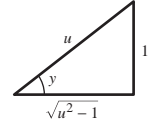
$$\frac{dy}{dx} = \frac{u'}{-\sin y} = -\frac{u'}{\sqrt{1-u^2}}$$


 (f) Let $y = \operatorname{arccsc} u$. Then

$$\csc y = u$$

$$-\csc y \cot y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}$$



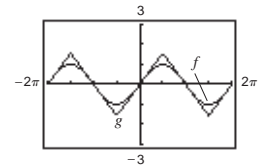
Note: The absolute value sign in the formula for the derivative of $\operatorname{arccsc} u$ is necessary because the inverse cosecant function has a negative slope at every value in its domain.

 96. $f(x) = \sin x$

$$g(x) = \arcsin(\sin x)$$

 (a) The range of $y = \arcsin x$ is $-\pi/2 \leq y \leq \pi/2$.

 (b) Maximum: $\pi/2$

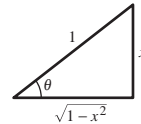
 Minimum: $-\pi/2$

 98. Let $\theta = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$, $-1 < x < 1$

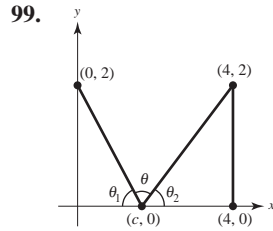
$$\tan \theta = \frac{x}{\sqrt{1-x^2}}$$

$$\sin \theta = \frac{x}{1} = x$$

$$\arcsin x = \theta.$$

Thus, $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$ for $-1 < x < 1$.





$$\tan \theta_1 = \frac{2}{c}, \quad \tan \theta_2 = \frac{2}{4-c}, \quad 0 < c < 4$$

To maximize θ , we minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{c}\right) + \arctan\left(\frac{2}{4-c}\right)$$

$$f'(c) = \frac{-2}{c^2+4} + \frac{2}{(4-c)^2+4} = 0$$

$$\frac{1}{c^2+4} = \frac{1}{(4-c)^2+4}$$

$$c^2+4 = c^2-8c+16+4$$

$$8c = 16$$

$$c = 2$$

By the First Derivative Test, $c = 2$ is a minimum. Hence, $(c, f(c)) = (2, \pi/2)$ is a relative maximum for the angle θ .
Checking the endpoints:

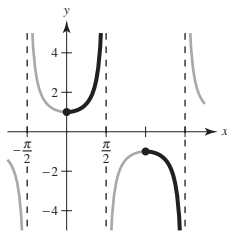
$$c = 0: \tan \theta = \frac{4}{2} = 2 \Rightarrow \theta \approx 1.107$$

$$c = 4: \tan \theta = \frac{4}{2} = 2 \Rightarrow \theta \approx 1.107$$

$$c = 2: \theta = \pi - \theta_1 - \theta_2 = \frac{\pi}{2} \approx 1.5708$$

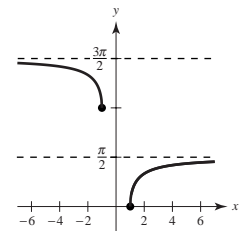
Thus, $(2, \pi/2)$ is the absolute maximum.

101. $f(x) = \sec x$, $0 \leq x < \frac{\pi}{2}$, $\pi \leq x < \frac{3\pi}{2}$



(a) $y = \operatorname{arcsec} x$, $x \leq -1$ or $x \geq 1$

$$0 \leq y < \frac{\pi}{2} \quad \text{or} \quad \pi \leq y < \frac{3\pi}{2}$$



(b) $y = \operatorname{arcsec} x$

$$x = \sec y$$

$$1 = \sec y \tan y \cdot y'$$

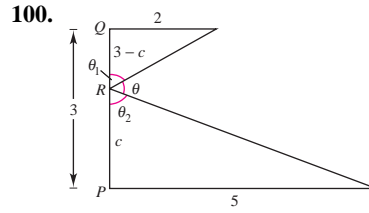
$$y' = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

$$\tan^2 y + 1 = \sec^2 y$$

$$\tan y = \pm \sqrt{\sec^2 y - 1}$$

On $0 \leq y < \pi/2$ and $\pi \leq y < 3\pi/2$, $\tan y \geq 0$.



$$\tan \theta_1 = \frac{2}{3-c}, \quad \tan \theta_2 = \frac{5}{c}, \quad 0 < c < 3$$

To maximize θ , minimize $f(c) = \theta_1 + \theta_2$.

$$f(c) = \arctan\left(\frac{2}{3-c}\right) + \arctan\left(\frac{5}{c}\right)$$

$$f'(c) = \frac{2}{(3-c)^2+4} + \frac{-5}{c^2+25} = 0$$

$$2(c^2+25) = 5(c^2-6c+9+4)$$

$$3c^2 - 30c + 15 = 0$$

$$c^2 - 10c + 5 = 0$$

$$c = 5 - 2\sqrt{5} \approx 0.5279 \quad (\text{since } c \in [0, 3])$$

$$\theta_1 + \theta_2 \approx 2.1458 \quad \text{and}$$

$$\theta \approx \pi - (\theta_1 + \theta_2) \approx 0.9958$$

Checking the endpoints:

$$c = 3 \Rightarrow \tan \theta = \frac{3}{5} \Rightarrow \theta \approx 0.5404$$

$$c = 0: \tan \theta = \frac{3}{2} \Rightarrow \theta \approx 0.9828$$

Thus, $c = 5 - 2\sqrt{5}$ yields the absolute maximum.

Section 5.7 Inverse Trigonometric Functions: Integration

1. $\int \frac{5}{\sqrt{9-x^2}} dx = 5 \arcsin\left(\frac{x}{3}\right) + C$
2. $\int \frac{3}{\sqrt{1-4x^2}} dx = \frac{3}{2} \int \frac{2}{\sqrt{1-4x^2}} dx = \frac{3}{2} \arcsin(2x) + C$
3. $\int \frac{7}{16+x^2} dx = \frac{7}{4} \arctan\left(\frac{x}{4}\right) + C$
4. $\int \frac{4}{1+9x^2} dx = \frac{4}{3} \int \frac{3}{1+9x^2} dx = \frac{4}{3} \arctan(3x) + C$
5. $\int \frac{1}{x\sqrt{4x^2-1}} dx = \int \frac{2}{2x\sqrt{(2x)^2-1}} dx = \operatorname{arcsec}|2x| + C$
6. $\int \frac{1}{4+(x-1)^2} dx = \frac{1}{2} \arctan\left(\frac{x-1}{2}\right) + C$
7. $\int \frac{x^3}{x^2+1} dx = \int \left[x - \frac{x}{x^2+1}\right] dx = \int x dx - \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2}x^2 - \frac{1}{2} \ln(x^2+1) + C$ (Use long division.)
8. $\int \frac{x^4-1}{x^2+1} dx = \int (x^2-1) dx = \frac{1}{3}x^3 - x + C$
9. $\int \frac{1}{\sqrt{1-(x+1)^2}} dx = \arcsin(x+1) + C$
10. Let $u = t^2$, $du = 2t dt$.

$$\int \frac{t}{t^4+16} dt = \frac{1}{2} \int \frac{1}{(4)^2 + (t^2)^2} (2t) dt = \frac{1}{8} \arctan \frac{t^2}{4} + C$$
11. Let $u = t^2$, $du = 2t dt$.

$$\int \frac{t}{\sqrt{1-t^4}} dt = \frac{1}{2} \int \frac{1}{\sqrt{1-(t^2)^2}} (2t) dt = \frac{1}{2} \arcsin(t^2) + C$$
12. Let $u = x^2$, $du = 2x dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^4-4}} dx &= \frac{1}{2} \int \frac{1}{x^2\sqrt{(x^2)^2-2^2}} (2x) dx \\ &= \frac{1}{4} \operatorname{arcsec} \frac{x^2}{2} + C \end{aligned}$$
13. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{4+e^{4x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{4+(e^{2x})^2} dx = \frac{1}{4} \arctan \frac{e^{2x}}{2} + C$$
14. $\int \frac{1}{3+(x-2)^2} dx = \int \frac{1}{(\sqrt{3})^2 + (x-2)^2} dx$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) + C$$
15. $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$, $u = \sqrt{x}$, $x = u^2$, $dx = 2u du$

$$\begin{aligned} \int \frac{1}{u\sqrt{1-u^2}} (2u du) &= 2 \int \frac{du}{\sqrt{1-u^2}} = 2 \arcsin u + C \\ &= 2 \arcsin \sqrt{x} + C \end{aligned}$$
16. $\int \frac{3}{2\sqrt{x}(1+x)} dx$, $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$, $dx = 2u du$

$$\begin{aligned} \frac{3}{2} \int \frac{2u du}{u(1+u^2)} &= 3 \int \frac{du}{1+u^2} = 3 \arctan u + C \\ &= 3 \arctan \sqrt{x} + C \end{aligned}$$
17. $\int \frac{x-3}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx - 3 \int \frac{1}{x^2+1} dx$

$$= \frac{1}{2} \ln(x^2+1) - 3 \arctan x + C$$
18. $\int \frac{4x+3}{\sqrt{1-x^2}} dx = (-2) \int \frac{-2x}{\sqrt{1-x^2}} dx + 3 \int \frac{1}{\sqrt{1-x^2}} dx = -4\sqrt{1-x^2} + 3 \arcsin x + C$
19. $\int \frac{x+5}{\sqrt{9-(x-3)^2}} dx = \int \frac{(x-3)}{\sqrt{9-(x-3)^2}} dx + \int \frac{8}{\sqrt{9-(x-3)^2}} dx$

$$= -\sqrt{9-(x-3)^2} - 8 \arcsin\left(\frac{x-3}{3}\right) + C = -\sqrt{6x-x^2} + 8 \arcsin\left(\frac{x}{3}-1\right) + C$$

$$\begin{aligned}
 20. \int \frac{x-2}{(x+1)^2+4} dx &= \frac{1}{2} \int \frac{2x+2}{(x+1)^2+4} dx - \int \frac{3}{(x+1)^2+4} dx \\
 &= \frac{1}{2} \ln(x^2+2x+5) - \frac{3}{2} \arctan\left(\frac{x+1}{2}\right) + C
 \end{aligned}$$

21. Let $u = 3x$, $du = 3 dx$.

$$\begin{aligned}
 \int_0^{1/6} \frac{1}{\sqrt{1-9x^2}} dx &= \frac{1}{3} \int_0^{1/6} \frac{1}{\sqrt{1-(3x)^2}} (3) dx \\
 &= \left[\frac{1}{3} \arcsin(3x) \right]_0^{1/6} = \frac{\pi}{18}
 \end{aligned}$$

23. Let $u = 2x$, $du = 2 dx$.

$$\begin{aligned}
 \int_0^{\sqrt{3}/2} \frac{1}{1+4x^2} dx &= \frac{1}{2} \int_0^{\sqrt{3}/2} \frac{2}{1+(2x)^2} dx \\
 &= \left[\frac{1}{2} \arctan(2x) \right]_0^{\sqrt{3}/2} = \frac{\pi}{6}
 \end{aligned}$$

25. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}} dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left[\frac{1}{2} \arcsin^2 x \right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

27. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned}
 \int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_{-1/2}^0 (1-x^2)^{-1/2} (-2x) dx \\
 &= \left[-\sqrt{1-x^2} \right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2} \\
 &\approx -0.134
 \end{aligned}$$

29. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned}
 \int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx &= -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x} dx \\
 &= \left[-\arctan(\cos x) \right]_{\pi/2}^{\pi} = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 31. \int_0^2 \frac{dx}{x^2-2x+2} &= \int_0^2 \frac{1}{1+(x-1)^2} dx \\
 &= \left[\arctan(x-1) \right]_0^2 = \frac{\pi}{2}
 \end{aligned}$$

$$22. \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$$

$$24. \int_{\sqrt{3}}^3 \frac{1}{9+x^2} dx = \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{\sqrt{3}}^3 = \frac{\pi}{36}$$

26. Let $u = \arccos x$, $du = -\frac{1}{\sqrt{1-x^2}} dx$.

$$\begin{aligned}
 \int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}} dx &= -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}} dx \\
 &= \left[-\frac{1}{2} \arccos^2 x \right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925
 \end{aligned}$$

28. Let $u = 1 + x^2$, $du = 2x dx$.

$$\begin{aligned}
 \int_{-\sqrt{3}}^0 \frac{x}{1+x^2} dx &= \frac{1}{2} \int_{-\sqrt{3}}^0 \frac{1}{1+x^2} (2x) dx \\
 &= \left[\frac{1}{2} \ln(1+x^2) \right]_{-\sqrt{3}}^0 = -\ln 2
 \end{aligned}$$

$$30. \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx = \arctan(\sin x) \Big|_0^{\pi/2} = \frac{\pi}{4}$$

$$\begin{aligned}
 32. \int_{-2}^2 \frac{dx}{x^2+4x+13} &= \int_{-2}^2 \frac{dx}{(x+2)^2+9} \\
 &= \left[\frac{1}{3} \arctan\left(\frac{x+2}{3}\right) \right]_{-2}^2 \\
 &= \frac{1}{3} \arctan\left(\frac{4}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 33. \int \frac{2x}{x^2 + 6x + 13} dx &= \int \frac{2x + 6}{x^2 + 6x + 13} dx - 6 \int \frac{1}{x^2 + 6x + 13} dx = \int \frac{2x + 6}{x^2 + 6x + 13} dx - 6 \int \frac{1}{4 + (x + 3)^2} dx \\
 &= \ln|x^2 + 6x + 13| - 3 \arctan\left(\frac{x + 3}{2}\right) + C
 \end{aligned}$$

$$34. \int \frac{2x - 5}{x^2 + 2x + 2} dx = \int \frac{2x + 2}{x^2 + 2x + 2} dx - 7 \int \frac{1}{1 + (x + 1)^2} dx = \ln|x^2 + 2x + 2| - 7 \arctan(x + 1) + C$$

$$\begin{aligned}
 35. \int \frac{1}{\sqrt{-x^2 - 4x}} dx &= \int \frac{1}{\sqrt{4 - (x + 2)^2}} dx \\
 &= \arcsin\left(\frac{x + 2}{2}\right) + C
 \end{aligned}$$

$$\begin{aligned}
 36. \int \frac{2}{\sqrt{-x^2 + 4x}} dx &= \int \frac{2}{\sqrt{4 - (x^2 - 4x + 4)}} dx \\
 &= \int \frac{2}{\sqrt{4 - (x - 2)^2}} dx \\
 &= 2 \arcsin\left(\frac{x - 2}{2}\right) + C
 \end{aligned}$$

$$37. \text{ Let } u = -x^2 - 4x, du = (-2x - 4) dx.$$

$$\begin{aligned}
 \int \frac{x + 2}{\sqrt{-x^2 - 4x}} dx &= -\frac{1}{2} \int (-x^2 - 4x)^{-1/2} (-2x - 4) dx \\
 &= -\sqrt{-x^2 - 4x} + C
 \end{aligned}$$

$$38. \text{ Let } u = x^2 - 2x, du = (2x - 2) dx.$$

$$\begin{aligned}
 \int \frac{x - 1}{\sqrt{x^2 - 2x}} dx &= \frac{1}{2} \int (x^2 - 2x)^{-1/2} (2x - 2) dx \\
 &= \sqrt{x^2 - 2x} + C
 \end{aligned}$$

$$\begin{aligned}
 39. \int_2^3 \frac{2x - 3}{\sqrt{4x - x^2}} dx &= \int_2^3 \frac{2x - 4}{\sqrt{4x - x^2}} dx + \int_2^3 \frac{1}{\sqrt{4x - x^2}} dx = -\int_2^3 (4x - x^2)^{-1/2} (4 - 2x) dx + \int_2^3 \frac{1}{\sqrt{4 - (x - 2)^2}} dx \\
 &= \left[-2\sqrt{4x - x^2} + \arcsin\left(\frac{x - 2}{2}\right) \right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059
 \end{aligned}$$

$$\begin{aligned}
 40. \int \frac{1}{(x - 1)\sqrt{x^2 - 2x}} dx &= \int \frac{1}{(x - 1)\sqrt{(x - 1)^2 - 1}} dx \\
 &= \operatorname{arcsec}|x - 1| + C
 \end{aligned}$$

$$41. \text{ Let } u = x^2 + 1, du = 2x dx.$$

$$\begin{aligned}
 \int \frac{x}{x^4 + 2x^2 + 2} dx &= \frac{1}{2} \int \frac{2x}{(x^2 + 1)^2 + 1} dx \\
 &= \frac{1}{2} \arctan(x^2 + 1) + C
 \end{aligned}$$

$$42. \text{ Let } u = x^2 - 4, du = 2x dx.$$

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}} dx = \frac{1}{2} \arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

$$43. \text{ Let } u = \sqrt{e^t - 3}. \text{ Then } u^2 + 3 = e^t, 2u du = e^t dt, \text{ and } \frac{2u du}{u^2 + 3} = dt.$$

$$\begin{aligned}
 \int \sqrt{e^t - 3} dt &= \int \frac{2u^2}{u^2 + 3} du = \int 2 du - \int 6 \frac{1}{u^2 + 3} du \\
 &= 2u - 2\sqrt{3} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3} \arctan \sqrt{\frac{e^t - 3}{3}} + C
 \end{aligned}$$

$$44. \text{ Let } u = \sqrt{x - 2}, u^2 + 2 = x, 2u du = dx.$$

$$\begin{aligned}
 \int \frac{\sqrt{x - 2}}{x + 1} dx &= \int \frac{2u^2}{u^2 + 3} du = \int \frac{2u^2 + 6 - 6}{u^2 + 3} du = 2 \int du - 6 \int \frac{1}{u^2 + 3} du \\
 &= 2u - \frac{6}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3} \arctan \sqrt{\frac{x - 2}{3}} + C
 \end{aligned}$$

$$45. \int_1^3 \frac{dx}{\sqrt{x}(1+x)}$$

Let $u = \sqrt{x}$, $u^2 = x$, $2u du = dx$, $1 + x = 1 + u^2$.

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{2u du}{u(1+u^2)} &= \int_1^{\sqrt{3}} \frac{2}{1+u^2} du \\ &= 2 \arctan(u) \Big|_1^{\sqrt{3}} \\ &= 2 \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\pi}{6} \end{aligned}$$

$$46. \int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$$

Let $u = \sqrt{x+1}$, $u^2 = x+1$, $2u du = dx$, $\sqrt{3-x} = \sqrt{4-u^2}$.

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{2u du}{2\sqrt{4-u^2}u} &= \int_1^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}} \\ &= \arcsin\left(\frac{u}{2}\right) \Big|_1^{\sqrt{2}} \\ &= \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right) \\ &= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \end{aligned}$$

$$47. (a) \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C, \quad u = x$$

$$(b) \int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C, \quad u = 1-x^2$$

(c) $\int \frac{1}{x\sqrt{1-x^2}} dx$ cannot be evaluated using the basic integration rules.

48. (a) $\int e^{x^2} dx$ cannot be evaluated using the basic integration rules.

$$(b) \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C, \quad u = x^2$$

$$(c) \int \frac{1}{x^2} e^{1/x} dx = -e^{1/x} + C, \quad u = \frac{1}{x}$$

$$49. (a) \int \sqrt{x-1} dx = \frac{2}{3}(x-1)^{3/2} + C, \quad u = x-1$$

(b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (u^2+1)(u)(2u) du = 2 \int (u^4 + u^2) du = 2 \left(\frac{u^5}{5} + \frac{u^3}{3} \right) + C \\ &= \frac{2}{15}u^3(3u^2+5) + C = \frac{2}{15}(x-1)^{3/2}[3(x-1)+5] + C = \frac{2}{15}(x-1)^{3/2}(3x+2) + C \end{aligned}$$

(c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u^2+1}{u}(2u) du = 2 \int (u^2+1) du = 2 \left(\frac{u^3}{3} + u \right) + C = \frac{2}{3}u(u^2+3) + C = \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

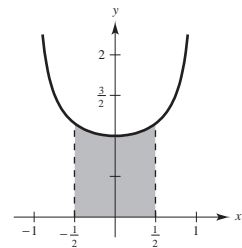
50. (a) $\int \frac{1}{1+x^4} dx$ cannot be evaluated using the basic integration rules.

$$\begin{aligned} (b) \int \frac{x}{1+x^4} dx &= \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx \\ &= \frac{1}{2} \arctan(x^2) + C, \quad u = x^2 \end{aligned}$$

$$\begin{aligned} (c) \int \frac{x^3}{1+x^4} dx &= \frac{1}{4} \int \frac{4x^3}{1+x^4} dx \\ &= \frac{1}{4} \ln(1+x^4) + C, \quad u = 1+x^4 \end{aligned}$$

51. Area $\approx (1)(1) = 1$

Matches (c)



52. No. This integral does not correspond to any of the basic differentiation rules.

53. $y' = \frac{1}{\sqrt{4-x^2}}, (0, \pi)$

$$y = \int \frac{1}{\sqrt{4-x^2}} dx = \arcsin\left(\frac{x}{2}\right) + C$$

$$y(0) = \pi = C$$

$$y = \arcsin\left(\frac{x}{2}\right) + \pi$$

54. $y' = \frac{1}{4+x^2}, (2, \pi)$

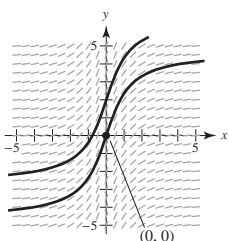
$$y = \int \frac{1}{4+x^2} dx = \frac{1}{2} \arctan \frac{x}{2} + C$$

$$\pi = \frac{1}{2} \arctan\left(\frac{2}{2}\right) + C$$

$$= \frac{\pi}{8} + C \Rightarrow C = \frac{7\pi}{8}$$

$$y = \frac{1}{2} \arctan\left(\frac{x}{2}\right) + \frac{7\pi}{8}$$

55. (a)

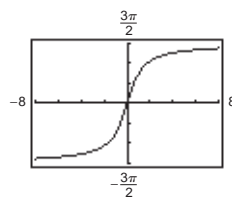


(b) $\frac{dy}{dx} = \frac{3}{1+x^2}, (0, 0)$

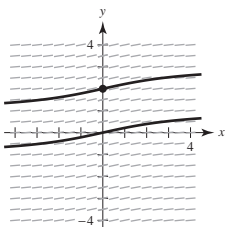
$$y = 3 \int \frac{dx}{1+x^2} = 3 \arctan x + C$$

$$(0, 0): 0 = 3 \arctan(0) + C \Rightarrow C = 0$$

$$y = 3 \arctan x$$



56. (a)

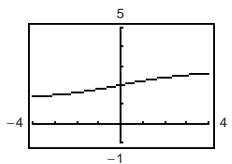


(b) $y' = \frac{2}{9+x^2}, (0, 2)$

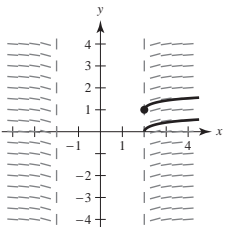
$$y = \int \frac{2}{9+x^2} dx = \frac{2}{3} \arctan\left(\frac{x}{3}\right) + C$$

$$2 = C$$

$$y = \frac{2}{3} \arctan\left(\frac{x}{3}\right) + 2$$



57. (a)

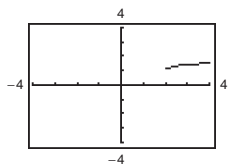


(b) $y' = \frac{1}{x\sqrt{x^2-4}}, (2, 1)$

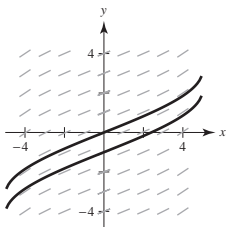
$$y = \int \frac{1}{x\sqrt{x^2-4}} dx = \frac{1}{2} \operatorname{arcsec} \frac{|x|}{2} + C$$

$$1 = \frac{1}{2} \operatorname{arcsec}(1) + C = C$$

$$y = \frac{1}{2} \operatorname{arcsec} \frac{x}{2} + 1, \quad x \geq 2$$



58. (a)

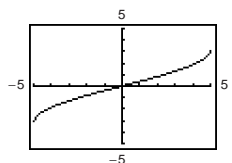


(b) $y' = \frac{2}{\sqrt{25-x^2}}, (5, \pi)$

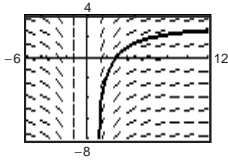
$$y = \int \frac{2}{\sqrt{25-x^2}} dx = 2 \arcsin\left(\frac{x}{5}\right) + C$$

$$\pi = 2 \arcsin(1) + C \Rightarrow C = 0$$

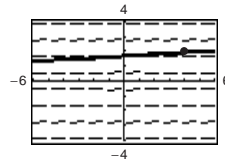
$$y = 2 \arcsin\left(\frac{x}{5}\right)$$



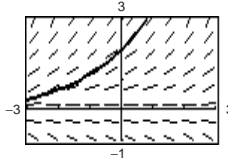
$$59. \frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}, \quad y(3) = 0$$



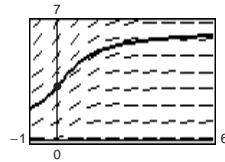
$$60. \frac{dy}{dx} = \frac{1}{12+x^2}, \quad y(4) = 2$$



$$61. \frac{dy}{dx} = \frac{2y}{\sqrt{16-x^2}}, \quad y(0) = 2$$



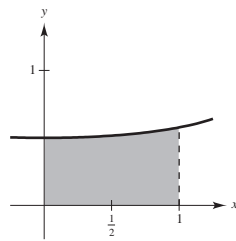
$$62. \frac{dy}{dx} = \frac{\sqrt{y}}{1+x^2}, \quad y(0) = 4$$



$$\begin{aligned} 63. A &= \int_1^3 \frac{1}{x^2 - 2x + 5} dx = \int_1^3 \frac{1}{(x-1)^2 + 4} dx \\ &= \frac{1}{2} \arctan\left(\frac{x-1}{2}\right) \Big|_1^3 \\ &= \frac{1}{2} \arctan(1) - \frac{1}{2} \arctan(0) \\ &= \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} 64. \text{Area} &= \int_{-2}^0 \frac{2}{x^2 + 4x + 8} dx = \int_{-2}^0 \frac{2}{(x+2)^2 + 4} dx \\ &= \arctan\left(\frac{x+2}{2}\right) \Big|_{-2}^0 \\ &= \arctan(1) - \arctan(0) \\ &= \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} 65. \text{Area} &= \int_0^1 \frac{1}{\sqrt{4-x^2}} dx \\ &= \arcsin\left(\frac{x}{2}\right) \Big|_0^1 \\ &= \arcsin\left(\frac{1}{2}\right) - \arcsin(0) \\ &= \frac{\pi}{6} \end{aligned}$$



$$\begin{aligned} 66. \text{Area} &= \int_{2/\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}} dx \\ &= \operatorname{arcsec} x \Big|_{2/\sqrt{2}}^2 \\ &= \operatorname{arcsec}(2) - \operatorname{arcsec}\left(\frac{2}{\sqrt{2}}\right) \\ &= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \end{aligned}$$

$$\begin{aligned} 67. \text{Area} &= \int_{-\pi/2}^{\pi/2} \frac{3 \cos x}{1 + \sin^2 x} dx = 3 \int_{-\pi/2}^{\pi/2} \frac{1}{1 + \sin^2 x} (\cos x dx) \\ &= 3 \arctan(\sin x) \Big|_{-\pi/2}^{\pi/2} \\ &= 3 \arctan(1) - 3 \arctan(-1) \\ &= \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2} \end{aligned}$$

$$\begin{aligned} 68. A &= \int_0^{\ln(\sqrt{3})} \frac{e^x}{1 + e^{2x}} dx, \quad (u = e^x) \\ &= \arctan(e^x) \Big|_0^{\ln \sqrt{3}} \\ &= \arctan(\sqrt{3}) - \arctan(1) \\ &= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \end{aligned}$$

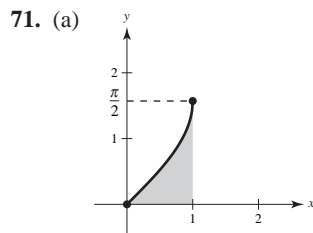
$$\begin{aligned}
 69. \text{ (a) } \frac{d}{dx} \left[\ln x - \frac{1}{2} \ln(1+x^2) - \frac{\arctan x}{x} + C \right] &= \frac{1}{x} - \frac{x}{1+x^2} - \left(\frac{x[1/(1+x^2)] - \arctan x}{x^2} \right) \\
 &= \frac{1+x^2-x^2}{x(1+x^2)} - \frac{1}{x(1+x^2)} + \frac{\arctan x}{x^2} = \frac{\arctan x}{x^2}
 \end{aligned}$$

$$\text{Thus, } \int \frac{\arctan x}{x^2} dx = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\arctan x}{x} + C.$$

$$\begin{aligned}
 \text{(b) } A &= \int_1^{\sqrt{3}} \frac{\arctan x}{x^2} dx \\
 &= \left[\ln x - \frac{1}{2} \ln(1+x^2) - \frac{\arctan x}{x} \right]_1^{\sqrt{3}} \\
 &= \left(\ln \sqrt{3} - \frac{1}{2} \ln(4) - \frac{\arctan \sqrt{3}}{\sqrt{3}} \right) - \left(-\frac{1}{2} \ln 2 - \arctan(1) \right) \\
 &= \frac{1}{2} \ln 3 - \frac{1}{2} \ln 2 - \frac{\pi\sqrt{3}}{9} + \frac{\pi}{4} \approx 0.3835
 \end{aligned}$$

$$\begin{aligned}
 70. \text{ (a) } \frac{d}{dx} [x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x + C] \\
 = (\arcsin x)^2 + 2x(\arcsin x) \frac{1}{\sqrt{1-x^2}} - 2 - \frac{2x}{\sqrt{1-x^2}} \arcsin x + 2\sqrt{1-x^2} \frac{1}{\sqrt{1-x^2}} = (\arcsin x)^2
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } A &= \int_0^1 (\arcsin x)^2 dx \\
 &= \left[x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x \right]_0^1 \\
 &= \left(\left(\frac{\pi}{2} \right)^2 - 2 \right) - (0) \\
 &= \frac{\pi^2}{4} - 2 \approx 0.4674
 \end{aligned}$$



Shaded area is given by $\int_0^1 \arcsin x dx$.

$$\text{(b) } \int_0^1 \arcsin x dx \approx 0.5708$$

(c) Divide the rectangle into two regions.

$$\text{Area rectangle} = (\text{base})(\text{height}) = 1 \left(\frac{\pi}{2} \right) = \frac{\pi}{2}$$

$$\text{Area rectangle} = \int_0^1 \arcsin x dx + \int_0^{\pi/2} \sin y dy$$

$$\frac{\pi}{2} = \int_0^1 \arcsin x dx + (-\cos y) \Big|_0^{\pi/2}$$

$$= \int_0^1 \arcsin x dx + 1$$

$$\text{Hence, } \int_0^1 \arcsin x dx = \frac{\pi}{2} - 1, (\approx 0.5708).$$

$$72. \text{ (a) } \int_0^1 \frac{4}{1+x^2} dx = \left[4 \arctan x \right]_0^1 = 4 \arctan 1 - 4 \arctan 0 = 4 \left(\frac{\pi}{4} \right) - 4(0) = \pi$$

(b) Let $n = 6$.

$$4 \int_0^1 \frac{1}{1+x^2} dx \approx 4 \left(\frac{1}{18} \right) \left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2} \right] \approx 3.1415918$$

(c) 3.1415927

$$73. F(x) = \frac{1}{2} \int_x^{x+2} \frac{2}{t^2 + 1} dt$$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

$$(b) F(x) = \left[\arctan t \right]_x^{x+2} = \arctan(x+2) - \arctan x$$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2) - (x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

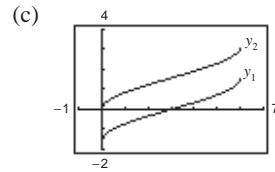
$$74. \int \frac{1}{\sqrt{6x-x^2}} dx$$

$$(a) 6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$$

$$\int \frac{1}{\sqrt{6x-x^2}} dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

$$(b) u = \sqrt{x}, u^2 = x, 2u du = dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{6u^2-u^4}} (2u du) &= \int \frac{2}{\sqrt{6-u^2}} du \\ &= 2 \arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2 \arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C \end{aligned}$$



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

$$75. \text{ False, } \int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{12} \operatorname{arcsec} \frac{|3x|}{4} + C$$

$$76. \text{ False, } \int \frac{dx}{25+x^2} dx = \frac{1}{5} \arctan \frac{x}{5} + C$$

77. True

$$\frac{d}{dx} \left[-\arccos \frac{x}{2} + C \right] = \frac{1/2}{\sqrt{1-(x/2)^2}} = \frac{1}{\sqrt{4-x^2}}$$

78. False. Use substitution: $u = 9 - e^{2x}, du = -2e^{2x} dx$

$$79. \frac{d}{dx} \left[\arcsin\left(\frac{u}{a}\right) + C \right] = \frac{1}{\sqrt{1-(u^2/a^2)}} \left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$$

$$\text{Thus, } \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$$

$$80. \frac{d}{dx} \left[\frac{1}{a} \arctan \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{1+(u/a)^2} \right]$$

$$= \frac{1}{a^2} \left[\frac{u'}{(a^2+u^2)/a^2} \right] = \frac{u'}{a^2+u^2}$$

$$\text{Thus, } \int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C.$$

81. Assume $u > 0$.

$$\frac{d}{dx} \left[\frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}} \right] = \frac{1}{a} \left[\frac{u'}{u\sqrt{(u^2-a^2)/a^2}} \right] = \frac{u'}{u\sqrt{u^2-a^2}}$$

The case $u < 0$ is handled in a similar manner. Thus,

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}} dx = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C.$$

$$82. (a) A = \int_0^1 \frac{1}{1+x^2} dx$$

(b) Trapezoidal Rule: $n = 8, b - a = 1 - 0 = 1$

$$A \approx 0.7847$$

—CONTINUED—

82. —CONTINUED—

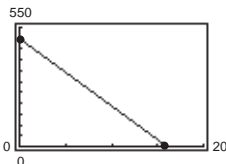
(c) Because

$$\int_0^1 \frac{1}{1+x^2} dx = \arctan x \Big|_0^1 = \frac{\pi}{4},$$

you can use the Trapezoidal Rule to approximate $\pi/4$, and hence, π . For example, using $n = 200$, you obtain

$$\pi \approx 4(0.785397) = 3.141588.$$

83. (a) $v(t) = -32t + 500$



$$\begin{aligned} \text{(c)} \quad \int \frac{1}{32 + kv^2} dv &= - \int dt \\ \frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}}v\right) &= -t + C_1 \\ \arctan\left(\sqrt{\frac{k}{32}}v\right) &= -\sqrt{32k}t + C \\ \sqrt{\frac{k}{32}}v &= \tan(C - \sqrt{32k}t) \\ v &= \sqrt{\frac{32}{k}} \tan(C - \sqrt{32k}t) \end{aligned}$$

When $t = 0$, $v = 500$, $C = \arctan(500\sqrt{k/32})$, and we have

$$v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}t\right].$$

$$\text{(e)} \quad h = \int_0^{6.86} \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032}t\right] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

84. Let $f(x) = \arctan x - \frac{x}{1+x^2}$

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$,

$$\arctan x - \frac{x}{1+x^2} > 0 \text{ for } x > 0. \text{ Thus, } \arctan x > \frac{x}{1+x^2}.$$

Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$. Therefore,

$$\frac{x}{1+x^2} < \arctan x < x.$$

$$\text{(b)} \quad s(t) = \int v(t) dt = \int (-32t + 500) dt$$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height, $v(t) = 0$.

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

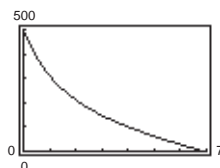
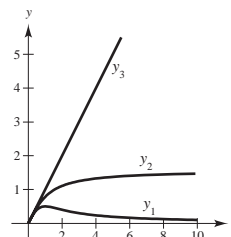
$$t = 15.625$$

$$s(15.625) = -16(15.625)^2 + 500(15.625)$$

$$= 3906.25 \text{ ft (Maximum height)}$$

(d) When $k = 0.001$:

$$v(t) = \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032}t\right]$$

 $v(t) = 0$ when $t_0 \approx 6.86$ sec.

Section 5.8 Hyperbolic Functions

$$1. (a) \sinh 3 = \frac{e^3 - e^{-3}}{2} \approx 10.018$$

$$(b) \tanh(-2) = \frac{\sinh(-2)}{\cosh(-2)} = \frac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$$

$$3. (a) \operatorname{csch}(\ln 2) = \frac{2}{e^{\ln 2} - e^{-\ln 2}} = \frac{2}{2 - (1/2)} = \frac{4}{3}$$

$$(b) \operatorname{coth}(\ln 5) = \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} \\ = \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}$$

$$5. (a) \cosh^{-1} 2 = \ln(2 + \sqrt{3}) \approx 1.317$$

$$(b) \operatorname{sech}^{-1} \frac{2}{3} = \ln\left(\frac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$$

$$2. (a) \cosh 0 = \frac{e^0 + e^0}{2} = 1$$

$$(b) \operatorname{sech} 1 = \frac{2}{e + e^{-1}} \approx 0.648$$

$$4. (a) \sinh^{-1} 0 = 0$$

$$(b) \tanh^{-1} 0 = 0$$

$$6. (a) \operatorname{csch}^{-1} 2 = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.481$$

$$(b) \operatorname{coth}^{-1} 3 = \frac{1}{2} \ln\left(\frac{4}{2}\right) \approx 0.347$$

$$7. \tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

$$8. \frac{1 + \cosh 2x}{2} = \frac{1 + (e^{2x} + e^{-2x})/2}{2} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$$

$$9. \sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ = \frac{1}{4}[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}] \\ = \frac{1}{4}[2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)$$

$$10. 2 \sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

$$11. 3 \sinh x + 4 \sinh^3 x = \sinh x(3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2}\right)\left[3 + 4\left(\frac{e^x - e^{-x}}{2}\right)^2\right] \\ = \left(\frac{e^x - e^{-x}}{2}\right)[3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2}(e^x - e^{-x})(e^{2x} + e^{-2x} + 1) \\ = \frac{1}{2}[e^{3x} + e^{-x} + e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)$$

$$12. 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2} = 2\left[\frac{e^{(x+y)/2} + e^{-(x+y)/2}}{2}\right]\left[\frac{e^{(x-y)/2} + e^{-(x-y)/2}}{2}\right] \\ = 2\left[\frac{e^x + e^y + e^{-y} + e^{-x}}{4}\right] = \frac{e^x + e^{-x}}{2} + \frac{e^y + e^{-y}}{2} \\ = \cosh x + \cosh y$$

$$13. \quad \sinh x = \frac{3}{2}$$

$$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\operatorname{coth} x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$14. \quad \tanh x = \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \Rightarrow \operatorname{sech}^2 x = \frac{3}{4} \Rightarrow \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\cosh x = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

$$\operatorname{coth} x = \frac{1}{1/2} = 2$$

$$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$$

$$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$$

Putting these in order:

$$\sinh x = \frac{\sqrt{3}}{3} \quad \operatorname{csch} x = \sqrt{3}$$

$$\cosh x = \frac{2\sqrt{3}}{3} \quad \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\tanh x = \frac{1}{2} \quad \operatorname{coth} x = 2$$

$$15. \quad y = \operatorname{sech}(x+1)$$

$$y' = -\operatorname{sech}(x+1) \tanh(x+1)$$

$$16. \quad y = \operatorname{coth} 3x$$

$$y' = -3 \operatorname{csch}^2(3x)$$

$$17. \quad f(x) = \ln(\sinh x)$$

$$f'(x) = \frac{1}{\sinh x} (\cosh x) = \operatorname{coth} x$$

$$18. \quad g(x) = \ln(\cosh x)$$

$$g'(x) = \frac{1}{\cosh x} (\sinh x) \\ = \tanh x$$

$$19. \quad y = \ln\left(\tanh \frac{x}{2}\right)$$

$$y' = \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) \\ = \frac{1}{2 \sinh(x/2) \cosh(x/2)} \\ = \frac{1}{\sinh x} = \operatorname{csch} x$$

$$20. \quad y = x \cosh x - \sinh x$$

$$y' = x \sinh x + \cosh x - \cosh x \\ = x \sinh x$$

$$21. \quad h(x) = \frac{1}{4} \sinh 2x - \frac{x}{2}$$

$$h'(x) = \frac{1}{2} \cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

$$22. \quad h(t) = t - \operatorname{coth} t$$

$$h'(t) = 1 + \operatorname{csch}^2 t = \operatorname{coth}^2 t$$

$$23. \quad f(t) = \arctan(\sinh t)$$

$$f'(t) = \frac{1}{1 + \sinh^2 t} (\cosh t) = \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

$$24. \quad g(x) = \operatorname{sech}^2 3x$$

$$g'(x) = -2 \operatorname{sech}(3x) \operatorname{sech}(3x) \tanh(3x)(3) \\ = -6 \operatorname{sech}^2 3x \tanh 3x$$

25. $y = \sinh(1 - x^2), (1, 0)$

$$y' = \cosh(1 - x^2)(-2x)$$

$$y'(1) = -2$$

Tangent line: $y - 0 = -2(x - 1)$

$$y = -2x + 2$$

27. $y = (\cosh x - \sinh x)^2, (0, 1)$

$$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$$

At $(0, 1), y' = 2(1)(-1) = -2$.

Tangent line: $y - 1 = -2(x - 0)$

$$y = -2x + 1$$

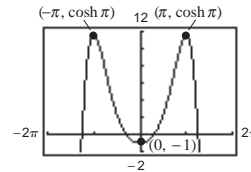
29. $f(x) = \sin x \sinh x - \cos x \cosh x, -4 \leq x \leq 4$

$$f'(x) = \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x$$

$$= 2 \sin x \cosh x = 0 \text{ when } x = 0, \pm\pi.$$

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$

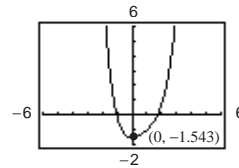


30. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

$$f'(x) = x \cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x \cosh(x - 1)$$

$$f'(x) = 0 \text{ for } x = 0.$$

By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



31. $g(x) = x \operatorname{sech} x$

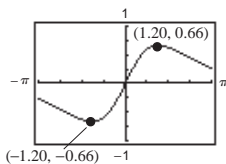
$$g'(x) = \operatorname{sech} x - x \operatorname{sech} x \tanh x$$

$$= \operatorname{sech} x(1 - x \tanh x) = 0$$

$$x \tanh x = 1$$

Using a graphing utility, $x \approx \pm 1.1997$.

By the First Derivative Test, $(1.1997, 0.6627)$ is a relative maximum and $(-1.1997, -0.6627)$ is a relative minimum.



26. $y = x^{\cosh x}, (1, 1)$

$$\ln y = \cosh x \ln x$$

$$\frac{y'}{y} = \frac{\cosh x}{x} + \sinh x \ln x$$

At $(1, 1), y' = \cosh(1)$.

Tangent line: $y - 1 = \cosh(1)(x - 1)$

$$y = \cosh(1)x - \cosh(1) + 1$$

Note: $\cosh(1) \approx 1.5431$

28. $y = e^{\sinh x}, (0, 1)$

$$y' = e^{\sinh x} \cosh x$$

$$y'(0) = e^0(1) = 1$$

Tangent line: $y - 1 = 1(x - 0)$

$$y = x + 1$$

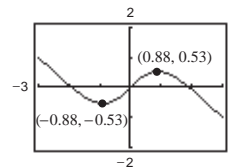
32. $h(x) = 2 \tanh x - x$

$$h'(x) = 2 \operatorname{sech}^2 x - 1 = 0$$

$$\operatorname{sech}^2 x = \frac{1}{2}$$

Using a graphing utility, $x \approx 0.8814$.

From the First Derivative Test, $(0.8814, 0.5328)$ is a relative maximum and $(-0.8814, -0.5328)$ is a relative minimum.



33. $y = a \sinh x$

$$y' = a \cosh x$$

$$y'' = a \sinh x$$

$$y''' = a \cosh x$$

Therefore, $y''' - y' = 0$.

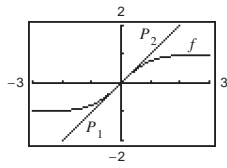
35. $f(x) = \tanh x, \quad f(0) = 0$

$$f'(x) = \operatorname{sech}^2 x, \quad f'(0) = 1$$

$$f''(x) = -2 \operatorname{sech}^2 x \tanh x, \quad f''(0) = 0$$

$$P_1(x) = f(0) + f'(0)(x - 0) = x$$

$$P_2(x) = f(0) + f'(0)(x - 0) + \frac{1}{2}f''(0)(x - 0)^2 = x$$



34. $y = a \cosh x$

$$y' = a \sinh x$$

$$y'' = a \cosh x$$

Therefore, $y'' - y = 0$.

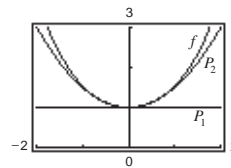
36. $f(x) = \cosh x \quad f(1) = \cosh(0) \approx 1$

$$f'(x) = \sinh x \quad f'(1) = \sinh(0) \approx 0$$

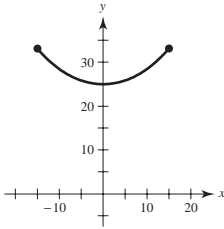
$$f''(x) = \cosh x \quad f''(1) = \cosh(0) \approx 1$$

$$P_1(x) = f(0) + f'(0)(x - 0) = 1$$

$$P_2(x) = 1 + \frac{1}{2}x^2$$



37. (a) $y = 10 + 15 \cosh \frac{x}{15}, \quad -15 \leq x \leq 15$

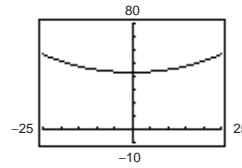


(b) At $x = \pm 15, y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0, y = 10 + 15 \cosh(0) = 25$.

(c) $y' = \sinh \frac{x}{15}$. At $x = 15, y' = \sinh(1) \approx 1.175$.

38. (a) $y = 18 + 25 \cosh \frac{x}{25}, \quad -25 \leq x \leq 25$



(b) At $x = \pm 25, y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0, y = 18 + 25 = 43$.

(c) $y' = \sinh \frac{x}{25}$. At $x = 25, y' = \sinh(1) \approx 1.175$.

39. Let $u = 1 - 2x, du = -2 dx$.

$$\begin{aligned} \int \sinh(1 - 2x) dx &= -\frac{1}{2} \int \sinh(1 - 2x)(-2) dx \\ &= -\frac{1}{2} \cosh(1 - 2x) + C \end{aligned}$$

41. Let $u = \cosh(x - 1), du = \sinh(x - 1) dx$.

$$\int \cosh^2(x - 1) \sinh(x - 1) dx = \frac{1}{3} \cosh^3(x - 1) + C$$

40. Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{\cosh \sqrt{x}}{\sqrt{x}} dx = 2 \int \cosh \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh \sqrt{x} + C$$

42. Let $u = \cosh x, du = \sinh x dx$.

$$\begin{aligned} \int \frac{\sinh x}{1 + \sinh^2 x} dx &= \int \frac{\sinh x}{\cosh^2 x} dx = \frac{-1}{\cosh x} + C \\ &= -\operatorname{sech} x + C \end{aligned}$$

43. Let $u = \sinh x$, $du = \cosh x dx$.

$$\int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2} dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2} \right) x dx = -\operatorname{coth} \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\begin{aligned} \int \frac{\operatorname{csch}(1/x) \operatorname{coth}(1/x)}{x^2} dx &= -\int \operatorname{csch} \frac{1}{x} \operatorname{coth} \frac{1}{x} \left(-\frac{1}{x^2} \right) dx \\ &= \operatorname{csch} \frac{1}{x} + C \end{aligned}$$

49. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{x^4 + 1} dx = \frac{1}{2} \int \frac{2x}{(x^2)^2 + 1} dx = \frac{1}{2} \arctan(x^2) + C$$

$$\begin{aligned} 51. \int_0^{\ln 2} \tanh x dx &= \int_0^{\ln 2} \frac{\sinh x}{\cosh x} dx, \quad (u = \cosh x) \\ &= \ln(\cosh x) \Big|_0^{\ln 2} \\ &= \ln(\cosh(\ln 2)) - \ln(\cosh(0)) \\ &= \ln\left(\frac{5}{4}\right) - 0 = \ln\left(\frac{5}{4}\right) \end{aligned}$$

$$\text{Note: } \cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + (1/2)}{2} = \frac{5}{4}$$

$$\begin{aligned} 53. \int_0^4 \frac{1}{25 - x^2} dx &= \frac{1}{10} \int \frac{1}{5 - x} dx + \frac{1}{10} \int \frac{1}{5 + x} dx \\ &= \left[\frac{1}{10} \ln \left| \frac{5+x}{5-x} \right| \right]_0^4 = \frac{1}{10} \ln 9 = \frac{1}{5} \ln 3 \end{aligned}$$

55. Let $u = 2x$, $du = 2 dx$.

$$\begin{aligned} \int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1-4x^2}} dx &= \int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1-(2x)^2}} (2) dx \\ &= \left[\arcsin(2x) \right]_0^{\sqrt{2}/4} = \frac{\pi}{4} \end{aligned}$$

44. Let $u = 2x - 1$, $du = 2 dx$.

$$\begin{aligned} \int \operatorname{sech}^2(2x - 1) dx &= \frac{1}{2} \int \operatorname{sech}^2(2x - 1) (2) dx \\ &= \frac{1}{2} \tanh(2x - 1) + C \end{aligned}$$

46. Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x dx$.

$$\begin{aligned} \int \operatorname{sech}^3 x \tanh x dx &= -\int \operatorname{sech}^2 x (-\operatorname{sech} x \tanh x) dx \\ &= -\frac{1}{3} \operatorname{sech}^3 x + C \end{aligned}$$

48. Let $u = \sinh x$, $du = \cosh x dx$.

$$\begin{aligned} \int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}} dx &= \arcsin\left(\frac{\sinh x}{3}\right) + C \\ &= \arcsin\left(\frac{e^x - e^{-x}}{6}\right) + C \end{aligned}$$

$$\begin{aligned} 50. \int \frac{2}{x\sqrt{1+4x^2}} dx &= 2 \int \frac{1}{(2x)\sqrt{1+(2x)^2}} (2) dx \\ &= 2 \ln\left(\frac{-1 + \sqrt{1+4x^2}}{|2x|}\right) + C \end{aligned}$$

$$\begin{aligned} 52. \int_0^1 \cosh^2 x dx &= \int_0^1 \frac{1 + \cosh(2x)}{2} dx \\ &= \frac{1}{2} \left[x + \frac{1}{2} \sinh(2x) \right]_0^1 \\ &= \frac{1}{2} \left[1 + \frac{1}{2} \sinh(2) \right] \\ &= \frac{1}{2} + \frac{1}{2} \sinh(1) \cosh(1) \end{aligned}$$

$$54. \int_0^4 \frac{1}{\sqrt{25-x^2}} dx = \left[\arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5}$$

$$56. \quad 2e^{-x} \cosh x = 2e^{-x} \left[\frac{e^x + e^{-x}}{2} \right] = 1 + e^{-2x}$$

$$\begin{aligned} \int_0^{\ln 2} 2e^{-x} \cosh x \, dx &= \int_0^{\ln 2} (1 + e^{-2x}) \, dx \\ &= \left[x - \frac{1}{2}e^{-2x} \right]_0^{\ln 2} \\ &= \left[\ln 2 - \frac{1}{2} \left(\frac{1}{4} \right) \right] - \left[0 - \frac{1}{2} \right] \\ &= \frac{3}{8} + \ln 2 \end{aligned}$$

$$57. \quad y = \cosh^{-1}(3x)$$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

$$58. \quad y = \tanh^{-1} \frac{x}{2}$$

$$y' = \frac{1}{1 - (x/2)^2} \left(\frac{1}{2} \right) = \frac{2}{4 - x^2}$$

$$59. \quad y = \sinh^{-1}(\tan x)$$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}} (\sec^2 x) = |\sec x|$$

$$60. \quad y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \frac{\pi}{4}$$

$$y' = \frac{-1}{\cos 2x \sqrt{1 - \cos^2 2x}} (-2 \sin 2x) = \frac{2 \sin 2x}{\cos 2x |\sin 2x|} = \frac{2}{\cos 2x} = 2 \sec 2x,$$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

$$61. \quad y = \tanh^{-1}(\sin 2x)$$

$$y' = \frac{1}{1 - \sin^2 2x} (2 \cos 2x) = 2 \sec 2x$$

$$62. \quad y = (\operatorname{csch}^{-1} x)^2$$

$$y' = 2 \operatorname{csch}^{-1} x \left(\frac{-1}{|x| \sqrt{1 + x^2}} \right) = \frac{-2 \operatorname{csch}^{-1} x}{|x| \sqrt{1 + x^2}}$$

$$63. \quad y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$$

$$\begin{aligned} y' &= 2x \left(\frac{2}{\sqrt{1 + 4x^2}} \right) + 2 \sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} \\ &= 2 \sinh^{-1}(2x) \end{aligned}$$

$$64. \quad y = x \tanh^{-1} x + \ln \sqrt{1 - x^2} = x \tanh^{-1} x + \frac{1}{2} \ln(1 - x^2)$$

$$y' = x \left(\frac{1}{1 - x^2} \right) + \tanh^{-1} x + \frac{-x}{1 - x^2} = \tanh^{-1} x$$

65. Answers will vary.

66. See the definitions and graphs in the textbook.

$$67. \quad \lim_{x \rightarrow \infty} \sinh x = \infty$$

$$68. \quad \lim_{x \rightarrow \infty} \tanh x = 1$$

$$69. \quad \lim_{x \rightarrow \infty} \operatorname{sech} x = 0$$

$$70. \quad \lim_{x \rightarrow -\infty} \operatorname{csch} x = 0$$

$$71. \quad \lim_{x \rightarrow 0} \frac{\sinh x}{x} = \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{2x} = 1$$

72. $\lim_{x \rightarrow 0} \operatorname{coth} x$ does not exist.

($\operatorname{coth} x \rightarrow \infty$ for $x \rightarrow 0^+$, $\operatorname{coth} x \rightarrow -\infty$ for $x \rightarrow 0^-$)

$$\begin{aligned} 73. \quad \int \frac{1}{\sqrt{1 + e^{2x}}} \, dx &= \int \frac{e^x}{e^x \sqrt{1 + (e^x)^2}} \, dx = -\operatorname{csch}^{-1}(e^x) + C \\ &= -\ln \left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x} \right) + C \end{aligned}$$

$$\begin{aligned} 74. \quad \int \frac{x}{9 - x^4} \, dx &= -\frac{1}{2} \int \frac{-2x}{9 - (x^2)^2} \, dx \\ &= -\frac{1}{2} \left(\frac{1}{6} \right) \ln \left| \frac{3 - x^2}{3 + x^2} \right| + C \\ &= -\frac{1}{12} \ln \left| \frac{3 - x^2}{3 + x^2} \right| + C \end{aligned}$$

75. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1+x}} dx = 2 \int \frac{1}{\sqrt{1+(\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}}\right) dx = 2 \sinh^{-1} \sqrt{x} + C = 2 \ln(\sqrt{x} + \sqrt{1+x}) + C$$

76. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x} dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx = \frac{2}{3} \int \frac{1}{\sqrt{1+(x^{3/2})^2}} \left(\frac{3}{2}\sqrt{x}\right) dx = \frac{2}{3} \sinh^{-1}(x^{3/2}) + C = \frac{2}{3} \ln(x^{3/2} + \sqrt{1+x^3}) + C$$

77. $\int \frac{-1}{4x-x^2} dx = \int \frac{1}{(x-2)^2-4} dx = \frac{1}{4} \ln \left| \frac{(x-2)-2}{(x-2)+2} \right| = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C$

78. $\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}} = -\frac{1}{2} \ln \left(\frac{2 + \sqrt{(x+2)^2+4}}{|x+2|} \right) + C$

79. $\int \frac{1}{1-4x-2x^2} dx = \int \frac{1}{3-2(x+1)^2} dx$
 $= \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}}{(\sqrt{3})^2 - [\sqrt{2}(x+1)]^2} dx$
 $= \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{3}} \ln \left| \frac{\sqrt{3} + \sqrt{2}(x+1)}{\sqrt{3} - \sqrt{2}(x+1)} \right| + C = \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) + \sqrt{3}}{\sqrt{2}(x+1) - \sqrt{3}} \right| + C$

80. $\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}} dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}} dx$
 $= \frac{1}{\sqrt{2}} \int \frac{1}{(x+1)\sqrt{(x+1)^2+(\sqrt{3})^2}} dx = -\frac{1}{\sqrt{6}} \ln \left(\frac{\sqrt{3} + \sqrt{(x+1)^2+3}}{x+1} \right) + C$

81. Let $u = 4x - 1$, $du = 4 dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}} dx$$

$$= \frac{1}{4} \int \frac{4}{\sqrt{81-(4x-1)^2}} dx = \frac{1}{4} \arcsin \left(\frac{4x-1}{9} \right) + C$$

82. Let $u = 2(x-1)$, $du = 2 dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}} dx$$

$$= \int \frac{2}{2(x-1)\sqrt{(\sqrt{3})^2 - [2(x-1)]^2}} dx$$

$$= -\frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{3} + \sqrt{-4x^2+8x-1}}{2(x-1)} \right| + C$$

83. $y = \int \frac{x^3-21x}{5+4x-x^2} dx = \int \left(-x-4 + \frac{20}{5+4x-x^2} \right) dx$
 $= \int (-x-4) dx + 20 \int \frac{1}{3^2 - (x-2)^2} dx$
 $= -\frac{x^2}{2} - 4x + \frac{20}{6} \ln \left| \frac{3+(x-2)}{3-(x-2)} \right| + C$
 $= -\frac{x^2}{2} - 4x + \frac{10}{3} \ln \left| \frac{1+x}{5-x} \right| + C$
 $= \frac{-x^2}{2} - 4x - \frac{10}{3} \ln \left| \frac{5-x}{x+1} \right| + C$

84. $y = \int \frac{1-2x}{4x-x^2} dx = \int \frac{4-2x}{4x-x^2} dx + 3 \int \frac{1}{(x-2)^2-4} dx$
 $= \ln|4x-x^2| + \frac{3}{4} \ln \left| \frac{(x-2)-2}{(x-2)+2} \right| + C$
 $= \ln|4x-x^2| + \frac{3}{4} \ln \left| \frac{x-4}{x} \right| + C$

$$\begin{aligned}
 85. A &= 2 \int_0^4 \operatorname{sech} \frac{x}{2} dx \\
 &= 2 \int_0^4 \frac{2}{e^{x/2} + e^{-x/2}} dx \\
 &= 4 \int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1} dx \\
 &= \left[8 \arctan(e^{x/2}) \right]_0^4 \\
 &= 8 \arctan(e^2) - 2\pi \approx 5.207
 \end{aligned}$$

$$\begin{aligned}
 87. A &= \int_0^2 \frac{5x}{\sqrt{x^4 + 1}} dx \\
 &= \frac{5}{2} \int_0^2 \frac{2x}{\sqrt{(x^2)^2 + 1}} dx \\
 &= \left[\frac{5}{2} \ln(x^2 + \sqrt{x^4 + 1}) \right]_0^2 \\
 &= \frac{5}{2} \ln(4 + \sqrt{17}) \approx 5.237
 \end{aligned}$$

$$\begin{aligned}
 89. (a) \int_0^{\sqrt{3}} \frac{dx}{\sqrt{x^2 + 1}} &= \ln(x + \sqrt{x^2 + 1}) \Big|_0^{\sqrt{3}} \\
 &= \ln(\sqrt{3} + 2) \approx 1.317
 \end{aligned}$$

$$\begin{aligned}
 90. (a) \int_{-1/2}^{1/2} \frac{dx}{1-x^2} &= \int_{-1/2}^{1/2} \left[\frac{1/2}{1+x} + \frac{1/2}{1-x} \right] dx \\
 &= \left[\frac{1}{2} \ln|1+x| - \frac{1}{2} \ln|1-x| \right]_{-1/2}^{1/2} \\
 &= \frac{1}{2} \left[\ln \frac{3}{2} - \ln \frac{1}{2} - \ln \frac{1}{2} + \ln \frac{3}{2} \right] \\
 &= \ln \frac{3}{2} - \ln \frac{1}{2} = \ln 3
 \end{aligned}$$

$$\begin{aligned}
 91. \int \frac{3k}{16} dt &= \int \frac{1}{x^2 - 12x + 32} dx \\
 \frac{3kt}{16} &= \int \frac{1}{(x-6)^2 - 4} dx = \frac{1}{2(2)} \ln \left| \frac{(x-6)-2}{(x-6)+2} \right| + C = \frac{1}{4} \ln \left| \frac{x-8}{x-4} \right| + C
 \end{aligned}$$

When $x = 0$:

$$t = 0$$

$$C = -\frac{1}{4} \ln(2)$$

When $x = 1$:

$$t = 10$$

$$\frac{30k}{16} = \frac{1}{4} \ln \left| \frac{-7}{-3} \right| - \frac{1}{4} \ln(2) = \frac{1}{4} \ln \left(\frac{7}{6} \right)$$

$$k = \frac{2}{15} \ln \left(\frac{7}{6} \right)$$

When $t = 20$:

$$\left(\frac{3}{16} \right) \left(\frac{2}{15} \right) \ln \left(\frac{7}{6} \right) (20) = \frac{1}{4} \ln \frac{x-8}{2x-8}$$

$$\ln \left(\frac{7}{6} \right)^2 = \ln \frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

$$\begin{aligned}
 86. A &= \int_0^2 \tanh 2x dx = \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx \\
 &= \frac{1}{2} \int_0^2 \frac{1}{e^{2x} + e^{-2x}} (2)(e^{2x} - e^{-2x}) dx \\
 &= \left[\frac{1}{2} \ln(e^{2x} + e^{-2x}) \right]_0^2 = \frac{1}{2} \ln(e^4 + e^{-4}) - \frac{1}{2} \ln 2 \\
 &= \ln \sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654
 \end{aligned}$$

$$\begin{aligned}
 88. A &= \int_3^5 \frac{6}{\sqrt{x^2 - 4}} dx \\
 &= \left[6 \ln(x + \sqrt{x^2 - 4}) \right]_3^5 \\
 &= 6 \ln(5 + \sqrt{21}) - 6 \ln(3 + \sqrt{5}) \\
 &= 6 \ln \left(\frac{5 + \sqrt{21}}{3 + \sqrt{5}} \right) \approx 3.626
 \end{aligned}$$

$$(b) \int_0^{\sqrt{3}} \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x \Big|_0^{\sqrt{3}} = \sinh^{-1}(\sqrt{3}) \approx 1.317$$

$$\begin{aligned}
 (b) \int_{-1/2}^{1/2} \frac{dx}{1-x^2} &= \left[\tanh^{-1} x \right]_{-1/2}^{1/2} \\
 &= \frac{\ln 3}{2} - \left(-\frac{\ln 3}{2} \right) = \ln 3
 \end{aligned}$$

92. (a) $v(t) = -32t$

(b) $s(t) = \int v(t) dt = \int (-32t) dt = -16t^2 + C$

$$s(0) = -16(0)^2 + C = 400 \Rightarrow C = 400$$

$$s(t) = -16t^2 + 400$$

(c) $\frac{dv}{dt} = -32 + kv^2$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

Let $u = \sqrt{k}v$, then $du = \sqrt{k} dv$.

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln \left| \frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} \right| = -t + C$$

Since $v(0) = 0$, $C = 0$.

$$\ln \left| \frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} \right| = -2\sqrt{32k}t$$

$$\frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} = e^{-2\sqrt{32k}t}$$

$$\sqrt{32} + \sqrt{k}v = e^{-2\sqrt{32k}t}(\sqrt{32} - \sqrt{k}v)$$

$$v(\sqrt{k} + \sqrt{k}e^{-2\sqrt{32k}t}) = \sqrt{32}(e^{-2\sqrt{32k}t} - 1)$$

$$v = \frac{\sqrt{32}(e^{-2\sqrt{32k}t} - 1)}{\sqrt{k}(e^{-2\sqrt{32k}t} + 1)} \cdot \frac{e^{\sqrt{32k}t}}{e^{\sqrt{32k}t}}$$

$$= \frac{\sqrt{32}}{\sqrt{k}} \left[\frac{-(e^{\sqrt{32k}t} - e^{-\sqrt{32k}t})}{e^{\sqrt{32k}t} + e^{-\sqrt{32k}t}} \right]$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t)$$

93. $y = a \operatorname{sech}^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}$, $a > 0$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

94. Equation of tangent line through $P = (x_0, y_0)$:

$$y - a \operatorname{sech}^{-1} \frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0}(x - x_0)$$

When $x = 0$,

$$y = a \operatorname{sech}^{-1} \frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a \operatorname{sech}^{-1} \frac{x_0}{a}$$

Hence, Q is the point $[0, a \operatorname{sech}^{-1}(x_0/a)]$.

Distance from P to Q : $d = \sqrt{x_0^2 + (-\sqrt{a^2 - x_0^2})^2} = a$

(d) $\lim_{t \rightarrow \infty} \left[-\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t) \right] = -\frac{\sqrt{32}}{\sqrt{k}}$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct) dt = (1/c) \ln \cosh(ct)$ (which can be verified by differentiation), then

$$\begin{aligned} s(t) &= \int -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t) dt \\ &= -\frac{\sqrt{32}}{\sqrt{k}} \frac{1}{\sqrt{32k}} \ln[\cosh(\sqrt{32k}t)] + C \\ &= -\frac{1}{k} \ln[\cosh(\sqrt{32k}t)] + C. \end{aligned}$$

When $t = 0$,

$$s(0) = C = 400 \Rightarrow 400 - (1/k) \ln[\cosh(\sqrt{32k}t)].$$

When $k = 0.01$:

$$s_2(t) = 400 - 100 \ln(\cosh \sqrt{0.32}t)$$

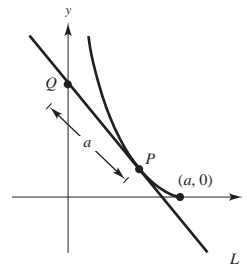
$$s_1(t) = -16t^2 + 400$$

$$s_1(t) = 0 \text{ when } t = 5 \text{ seconds}$$

$$s_2(t) = 0 \text{ when } t \approx 8.3 \text{ seconds}$$

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

(f) As k increases, the time required for the object to reach the ground increases.



95. Let $u = \tanh^{-1} x$, $-1 < x < 1$

$$\tanh u = x.$$

$$\frac{\sinh u}{\cosh u} = \frac{e^u - e^{-u}}{e^u + e^{-u}} = x$$

$$e^u - e^{-u} = xe^u + xe^{-u}$$

$$e^{2u} - 1 = xe^{2u} + x$$

$$e^{2u}(1 - x) = 1 + x$$

$$e^{2u} = \frac{1 + x}{1 - x}$$

$$2u = \ln\left(\frac{1 + x}{1 - x}\right)$$

$$u = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right), \quad -1 < x < 1$$

97.
$$\int_{-b}^b e^{xt} dt = \left[\frac{e^{xt}}{x}\right]_{-b}^b$$

$$= \frac{e^{xb}}{x} - \frac{e^{-xb}}{x}$$

$$= \frac{2}{x} \left[\frac{e^{xb} - e^{-xb}}{2} \right]$$

$$= \frac{2}{x} \sinh(xb)$$

99. $y = \operatorname{sech}^{-1} x$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)}$$

$$= \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

101. $y = \sinh^{-1} x$

$$\sinh y = x$$

$$(\cosh y)y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

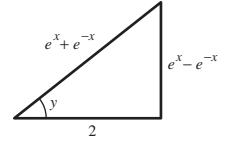
96. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$



98. $y = \cosh x = \frac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

100. $y = \cosh^{-1} x$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

102. $y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x})$$

$$= \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

$$103. y = c \cosh \frac{x}{c}$$

Let $P(x_1, y_1)$ be a point on the catenary.

$$y' = \sinh \frac{x}{c}$$

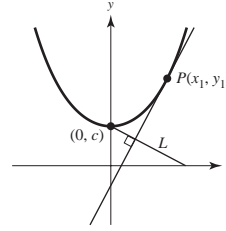
The slope at P is $\sinh(x_1/c)$. The equation of line L is

$$y - c = \frac{-1}{\sinh(x_1/c)}(x - 0).$$

When $y = 0$, $c = \frac{x}{\sinh(x_1/c)} \Rightarrow x = c \sinh\left(\frac{x_1}{c}\right)$. The length of L is

$$\sqrt{c^2 \sinh^2\left(\frac{x_1}{c}\right) + c^2} = c \cdot \cosh \frac{x_1}{c} = y_1,$$

the ordinate y_1 of the point P .



104. There is no such common normal. To see this, assume there is a common normal.

$$y = \cosh x \Rightarrow y' = \sinh x.$$

$$\text{Normal line at } (a, \cosh a) \text{ is } y - \cosh a = \frac{-1}{\sinh a}(x - a).$$

Similarly, $y - \sinh c = \frac{-1}{\cosh c}(x - c)$ is normal at $(c, \sinh c)$. Also,

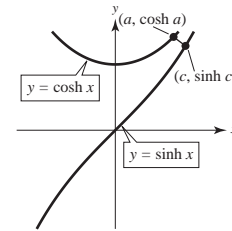
$$\frac{-1}{\sinh a} = \frac{-1}{\cosh c} \Rightarrow \cosh c = \sinh a.$$

The slope between the points is $\frac{\sinh c - \cosh a}{c - a}$. Therefore, $-\frac{a - c}{\cosh a - \sinh c} = \cosh c = \sinh a$.

$$\cosh c > 0 \Rightarrow a > 0$$

$\sinh x < \cosh x$ for all $x \Rightarrow \sinh c < \cosh c = \sinh a < \cosh a$. Hence, $c < a$. But,

$$-\frac{a - c}{\cosh a - \sinh c} < 0, \text{ a contradiction.}$$

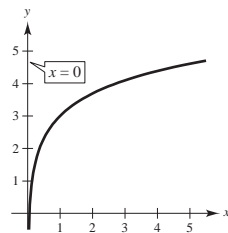


Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward

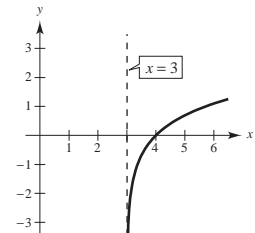
Vertical asymptote: $x = 0$



2. $f(x) = \ln(x - 3)$

Horizontal shift 3 units to the right

Vertical asymptote: $x = 3$



3. $\ln \sqrt{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$

4. $\ln[(x^2 + 1)(x - 1)] = \ln(x^2 + 1) + \ln(x - 1)$

5. $\ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x$
 $= \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$

$$6. 3[\ln x - 2 \ln(x^2 + 1)] + 2 \ln 5 = 3 \ln x - 6 \ln(x^2 + 1) + \ln 5^2$$

$$= \ln x^3 - \ln(x^2 + 1)^6 + \ln 25 = \ln \left[\frac{25x^3}{(x^2 + 1)^6} \right]$$

$$7. \ln \sqrt{x+1} = 2$$

$$\sqrt{x+1} = e^2$$

$$x+1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$8. \ln x + \ln(x-3) = 0$$

$$\ln x(x-3) = 0$$

$$x(x-3) = e^0$$

$$x^2 - 3x - 1 = 0$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

$$x = \frac{3 + \sqrt{13}}{2} \text{ only since } \frac{3 - \sqrt{13}}{2} < 0.$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$10. h(x) = \ln \frac{x(x-1)}{x-2} = \ln x + \ln(x-1) - \ln(x-2)$$

$$h'(x) = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x-2} = \frac{x^2 - 4x + 2}{x^3 - 3x^2 + 2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2} \left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2 \ln x}{2\sqrt{\ln x}}$$

$$12. f(x) = \ln[x(x^2 - 2)^{2/3}] = \ln x + \frac{2}{3} \ln(x^2 - 2)$$

$$f'(x) = \frac{1}{x} + \frac{2}{3} \left(\frac{2x}{x^2 - 2}\right) = \frac{7x^2 - 6}{3x^3 - 6x}$$

$$13. y = \frac{1}{b^2} [a + bx - a \ln(a + bx)]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left(b - \frac{ab}{a + bx} \right) = \frac{x}{a + bx}$$

$$14. y = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{a + bx}{x}$$

$$= -\frac{1}{ax} + \frac{b}{a^2} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(-\frac{1}{x^2} \right) + \frac{b}{a^2} \left[\frac{b}{a + bx} - \frac{1}{x} \right]$$

$$= \frac{1}{ax^2} + \frac{b}{a^2} \left[\frac{-a}{x(a + bx)} \right] = \frac{1}{ax^2} - \frac{b}{ax(a + bx)}$$

$$= \frac{(a + bx) - bx}{ax^2(a + bx)} = \frac{1}{x^2(a + bx)}$$

$$15. y = \ln(2 + x) + \frac{2}{2 + x}, \quad (-1, 2)$$

$$y' = \frac{1}{2 + x} - \frac{2}{(2 + x)^2}$$

$$y'(-1) = 1 - 2 = -1$$

$$\text{Tangent line: } y - 2 = -1(x + 1)$$

$$y = -x + 1$$

$$16. y = \ln \frac{1 + x}{x} = \ln(1 + x) - \ln x, \quad (1, \ln 2)$$

$$y' = \frac{1}{1 + x} - \frac{1}{x}$$

$$y'(1) = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\text{Tangent line: } y - \ln 2 = -\frac{1}{2}(x - 1)$$

$$y = -\frac{1}{2}x + \ln 2 + \frac{1}{2}$$

17. $u = 7x - 2, du = 7 dx$

$$\int \frac{1}{7x-2} dx = \frac{1}{7} \int \frac{1}{7x-2} (7) dx = \frac{1}{7} \ln|7x-2| + C$$

19.
$$\int \frac{\sin x}{1 + \cos x} dx = - \int \frac{-\sin x}{1 + \cos x} dx$$

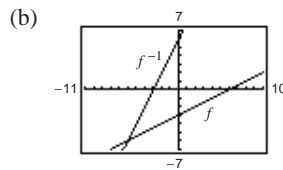
$$= -\ln|1 + \cos x| + C$$

21.
$$\int_1^4 \frac{x+1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x|\right]_1^4$$

$$= 3 + \ln 4$$

23.
$$\int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta|\right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

25. (a) $f(x) = \frac{1}{2}x - 3$
 $y = \frac{1}{2}x - 3$
 $2(y + 3) = x$
 $2(x + 3) = y$
 $f^{-1}(x) = 2x + 6$



18. $u = x^2 - 1, du = 2x dx$

$$\int \frac{x}{x^2-1} dx = \frac{1}{2} \int \frac{2x}{x^2-1} dx = \frac{1}{2} \ln|x^2-1| + C$$

20. $u = \ln x, du = \frac{1}{x} dx$

$$\int \frac{\ln \sqrt{x}}{x} dx = \frac{1}{2} \int (\ln x) \left(\frac{1}{x}\right) dx = \frac{1}{4} (\ln x)^2 + C$$

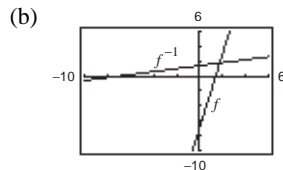
22.
$$\int_1^e \frac{\ln x}{x} dx = \int_1^e (\ln x) \left(\frac{1}{x}\right) dx = \left[\frac{1}{2} (\ln x)^2\right]_1^e = \frac{1}{2}$$

24.
$$\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx = \left[\ln \left|\cos\left(\frac{\pi}{4} - x\right)\right|\right]_0^{\pi/4}$$

$$= 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \ln 2$$

(c) $f^{-1}(f(x)) = f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x$
 $f(f^{-1}(x)) = f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x$

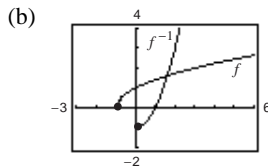
26. (a) $f(x) = 5x - 7$
 $y = 5x - 7$
 $\frac{y+7}{5} = x$
 $\frac{x+7}{5} = y$
 $f^{-1}(x) = \frac{x+7}{5}$



(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \frac{(5x - 7) + 7}{5} = x$

$f(f^{-1}(x)) = f\left(\frac{x+7}{5}\right) = 5\left(\frac{x+7}{5}\right) - 7 = x$

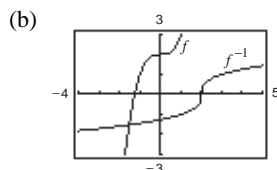
27. (a) $f(x) = \sqrt{x+1}$
 $y = \sqrt{x+1}$
 $y^2 - 1 = x$
 $x^2 - 1 = y$
 $f^{-1}(x) = x^2 - 1, x \geq 0$



(c) $f^{-1}(f(x)) = f^{-1}(\sqrt{x+1}) = \sqrt{(x^2 - 1)^2} - 1 = x$

$f(f^{-1}(x)) = f(x^2 - 1) = \sqrt{(x^2 - 1) + 1}$
 $= \sqrt{x^2} = x \text{ for } x \geq 0.$

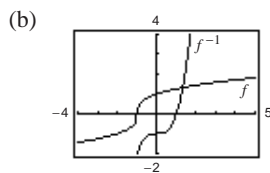
28. (a) $f(x) = x^3 + 2$
 $y = x^3 + 2$
 $\sqrt[3]{y-2} = x$
 $\sqrt[3]{x-2} = y$
 $f^{-1}(x) = \sqrt[3]{x-2}$



(c) $f^{-1}(f(x)) = f^{-1}(x^3 + 2) = \sqrt[3]{(x^3 + 2) - 2} = x$

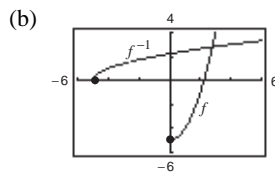
$f(f^{-1}(x)) = f(\sqrt[3]{x-2}) = (\sqrt[3]{x-2})^3 + 2 = x$

29. (a) $f(x) = \sqrt[3]{x+1}$
 $y = \sqrt[3]{x+1}$
 $y^3 - 1 = x$
 $x^3 - 1 = y$
 $f^{-1}(x) = x^3 - 1$



(c) $f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x+1})$
 $= (\sqrt[3]{x+1})^3 - 1 = x$
 $f(f^{-1}(x)) = f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x$

30. (a) $f(x) = x^2 - 5, x \geq 0$
 $y = x^2 - 5$
 $\sqrt{y+5} = x$
 $\sqrt{x+5} = y$
 $f^{-1}(x) = \sqrt{x+5}$



(c) $f^{-1}(f(x)) = f^{-1}(x^2 - 5)$
 $= \sqrt{(x^2 - 5) + 5} = x$ for $x \geq 0$.
 $f(f^{-1}(x)) = f(\sqrt{x+5}) = (\sqrt{x+5})^2 - 5 = x$

31. $f(x) = x^3 + 2$
 $f^{-1}(x) = (x-2)^{1/3}$
 $(f^{-1})'(x) = \frac{1}{3}(x-2)^{-2/3}$
 $(f^{-1})'(-1) = \frac{1}{3}(-1-2)^{-2/3} = \frac{1}{3(-3)^{2/3}}$
 $= \frac{1}{3^{5/3}} \approx 0.160$

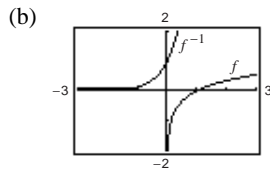
or $f(-3^{1/3}) = -1$
 $f'(x) = 3x^2$
 $f'(-3^{1/3}) = 3(-3^{1/3})^2 = 3^{5/3}$
 $(f^{-1})'(-1) = \frac{1}{f'(-3^{1/3})} = \frac{1}{3^{5/3}}$

32. $f(x) = x\sqrt{x-3}$
 $f(4) = 4$
 $f'(x) = \sqrt{x-3} + \frac{1}{2}x(x-3)^{-1/2}$
 $f'(4) = 1 + 2 = 3$
 $(f^{-1})'(4) = \frac{1}{f'(4)} = \frac{1}{3}$

33. $f(x) = \tan x$
 $f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$
 $f'(x) = \sec^2 x$
 $f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$
 $(f^{-1})'\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'(\pi/6)} = \frac{3}{4}$

34. $f(x) = \ln x$
 $f^{-1}(x) = e^x$
 $(f^{-1})'(x) = e^x$
 $(f^{-1})'(0) = e^0 = 1$

35. (a) $f(x) = \ln \sqrt{x}$
 $y = \ln \sqrt{x}$
 $e^y = \sqrt{x}$
 $e^{2y} = x$
 $e^{2x} = y$
 $f^{-1}(x) = e^{2x}$



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$
 $f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$

36. (a) $f(x) = e^{1-x}$

$$y = e^{1-x}$$

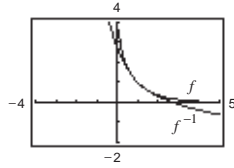
$$\ln y = 1 - x$$

$$x = 1 - \ln y$$

$$y = 1 - \ln x$$

$$f^{-1}(x) = 1 - \ln x$$

(b)

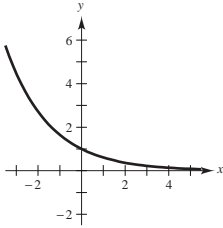


(c) $f^{-1}(f(x)) = f^{-1}(e^{1-x}) = 1 - \ln(e^{1-x})$

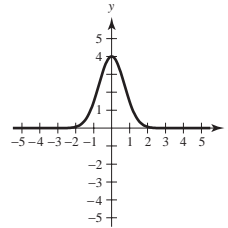
$$= 1 - (1 - x) = x$$

$$f(f^{-1}(x)) = f(1 - \ln x) = e^{1 - (1 - \ln x)} = e^{\ln x} = x$$

37. $y = e^{-x/2}$



38. $y = 4e^{-x^2}$



39. $g(t) = t^2 e^t$

$$g'(t) = t^2 e^t + 2te^t = te^t(t + 2)$$

40. $g(x) = \ln\left(\frac{e^x}{1 + e^x}\right)$

$$= \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x)$$

$$g'(x) = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

41. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

42. $h(z) = e^{-z^2/2}$

$$h'(z) = -ze^{-z^2/2}$$

43. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

44. $y = 3e^{-3/t}$

$$y' = 3e^{-3/t}(3t^{-2}) = \frac{9e^{-3/t}}{t^2}$$

45. $f(x) = \ln(e^{-x^2}) = -x^2, \quad (2, -4)$

$$f'(x) = -2x$$

$$f'(2) = -4$$

Tangent line: $y + 4 = -4(x - 2)$

$$y = -4x + 4$$

46. $f(\theta) = \frac{1}{2}e^{\sin 2\theta}, \quad (0, \frac{1}{2})$

$$f'(\theta) = \frac{1}{2}e^{\sin 2\theta} \cdot 2 \cos 2\theta$$

$$f'(0) = 1$$

Tangent line: $y - \frac{1}{2} = 1(x - 0)$

$$y = x + \frac{1}{2}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

48. $\cos x^2 = xe^y$

$$-2x \sin x^2 = xe^y \frac{dy}{dx} + e^y$$

$$\frac{dy}{dx} = -\frac{2x \sin x^2 + e^y}{xe^y}$$

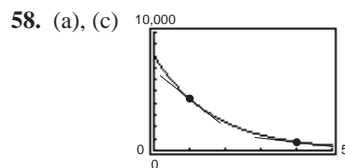
$$\begin{aligned}
 49. \int_0^1 x e^{-3x^2} dx &= -\frac{1}{6} \int_0^1 e^{-3x^2} (-6x dx) \\
 &= -\frac{1}{6} e^{-3x^2} \Big|_0^1 \\
 &= -\frac{1}{6} [e^{-3} - 1] \\
 &= \frac{1}{6} \left(1 - \frac{1}{e^3} \right)
 \end{aligned}$$

$$\begin{aligned}
 51. \int \frac{e^{4x} - e^{2x} + 1}{e^x} dx &= \int (e^{3x} - e^x + e^{-x}) dx \\
 &= \frac{1}{3} e^{3x} - e^x - e^{-x} + C \\
 &= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C
 \end{aligned}$$

$$\begin{aligned}
 53. \int x e^{1-x^2} dx &= -\frac{1}{2} \int e^{1-x^2} (-2x) dx \\
 &= -\frac{1}{2} e^{1-x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 55. \int_1^3 \frac{e^x}{e^x - 1} dx \\
 \text{Let } u = e^x - 1, du = e^x dx. \\
 \int_1^3 \frac{e^x}{e^x - 1} dx &= \ln|e^x - 1| \Big|_1^3 \\
 &= \ln(e^3 - 1) - \ln(e - 1) \\
 &= \ln\left(\frac{e^3 - 1}{e - 1}\right) \\
 &= \ln(e^2 + e + 1)
 \end{aligned}$$

$$\begin{aligned}
 57. \quad y &= e^x(a \cos 3x + b \sin 3x) \\
 y' &= e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x) \\
 &= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x] \\
 y'' &= e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x] \\
 &= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x] \\
 y'' - 2y' + 10y &= e^x\{[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x\} = 0
 \end{aligned}$$



$$\begin{aligned}
 50. \int_{1/2}^2 \frac{e^{1/x}}{x^2} dx \\
 \text{Let } u = \frac{1}{x}, du = \frac{-1}{x^2} dx. \\
 x = \frac{1}{2} \Rightarrow u = 2, \quad x = 2 \Rightarrow u = \frac{1}{2} \\
 -\int_2^{1/2} e^u du = -e^u \Big|_2^{1/2} = -e^{1/2} + e^2 = e^2 - \sqrt{e}
 \end{aligned}$$

$$\begin{aligned}
 52. \text{ Let } u = e^{2x} + e^{-2x}, du = (2e^{2x} - e^{-2x}) dx. \\
 \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx &= \frac{1}{2} \int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}} dx \\
 &= \frac{1}{2} \ln(e^{2x} + e^{-2x}) + C
 \end{aligned}$$

$$\begin{aligned}
 54. \text{ Let } u = x^3 + 1, du = 3x^2 dx. \\
 \int x^2 e^{x^3+1} dx &= \frac{1}{3} \int e^{x^3+1} (3x^2) dx = \frac{1}{3} e^{x^3+1} + C
 \end{aligned}$$

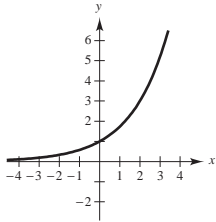
$$\begin{aligned}
 56. \int_0^2 \frac{e^{2x}}{e^{2x} + 1} dx \\
 \text{Let } u = e^{2x} + 1, du = 2e^{2x} dx. \\
 \frac{1}{2} \int_0^2 \frac{2e^{2x}}{e^{2x} + 1} dx &= \frac{1}{2} \ln(e^{2x} + 1) \Big|_0^2 \\
 &= \frac{1}{2} \ln(e^4 + 1) - \frac{1}{2} \ln 2 \\
 &= \frac{1}{2} \ln\left(\frac{e^4 + 1}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad V &= 8000e^{-0.6t}, \quad 0 \leq t \leq 5 \\
 V'(t) &= -4800e^{-0.6t} \\
 V'(1) &= -2634.3 \text{ dollars/year} \\
 V'(4) &= -435.4 \text{ dollars/year}
 \end{aligned}$$

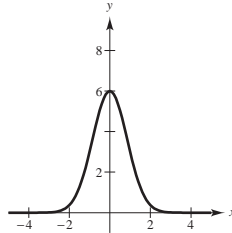
$$59. \text{Area} = \int_0^4 xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2} \right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$$

$$60. \text{Area} = \int_0^2 2e^{-x} dx = \left[-2e^{-x} \right]_0^2 = -2e^{-2} + 2 = 2 - \frac{2}{e^2} \approx 1.729$$

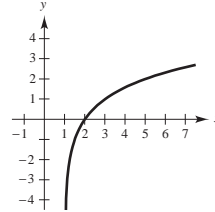
$$61. y = 3^{x/2}$$



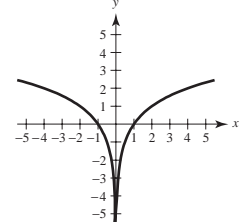
$$62. g(x) = 6(2^{-x^2})$$



$$63. y = \log_2(x - 1)$$



$$64. y = \log_4 x^2$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$66. f(x) = 4^x e^x$$

$$f'(x) = 4^x e^x + (\ln 4)4^x e^x \\ = 4^x e^x (1 + \ln 4)$$

$$67. y = x^{2x+1}$$

$$\ln y = (2x + 1) \ln x$$

$$\frac{y'}{y} = \frac{2x + 1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x + 1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x + 1}{x} + 2 \ln x \right)$$

$$68. y = x(4^{-x})$$

$$y' = 4^{-x} - x \cdot 4^{-x} \ln 4$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \left(\frac{1}{2} \right) \frac{-1}{(1-x) \ln 3} = \frac{-1}{2(x-1) \ln 3}$$

$$70. h(x) = \log_5 \frac{x}{x-1} = \log_5 x - \log_5(x-1)$$

$$h'(x) = \frac{1}{\ln 5} \left[\frac{1}{x} - \frac{1}{x-1} \right] = \frac{1}{\ln 5} \left[\frac{-1}{x(x-1)} \right]$$

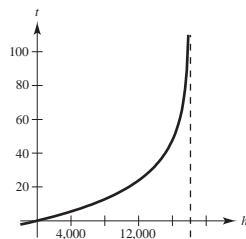
$$71. \int (x+1)5^{(x+1)^2} dx = \left(\frac{1}{2} \right) \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$72. \int \frac{2^{-1/t}}{t^2} dt = \frac{1}{\ln 2} 2^{-1/t} + C$$

$$73. t = 50 \log_{10} \left(\frac{18,000}{18,000 - h} \right)$$

$$(a) \text{Domain: } 0 \leq h < 18,000$$

(b)



Vertical asymptote: $h = 18,000$

$$(c) t = 50 \log_{10} \left(\frac{18,000}{18,000 - h} \right)$$

$$10^{t/50} = \frac{18,000}{18,000 - h}$$

$$18,000 - h = 18,000(10^{-t/50})$$

$$h = 18,000(1 - 10^{-t/50})$$

$$\frac{dh}{dt} = 360 \ln 10 \left(\frac{1}{10} \right)^{t/50} \text{ is greatest when } t = 0.$$

74. (a) $10,000 = Pe^{(0.07)(15)}$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

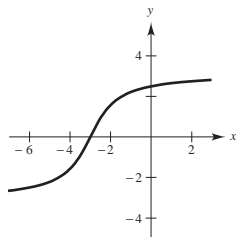
(b) $2P = Pe^{10r}$

$2 = e^{10r}$

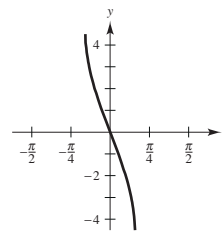
$\ln 2 = 10r$

$$r = \frac{\ln 2}{10} \approx 6.93\%$$

75. $f(x) = 2 \arctan(x + 3)$



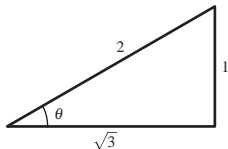
76. $h(x) = -3 \arcsin(2x)$



77. (a) Let $\theta = \arcsin \frac{1}{2}$

$\sin \theta = \frac{1}{2}$

$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}$



(b) Let $\theta = \arcsin \frac{1}{2}$

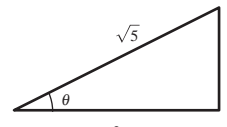
$\sin \theta = \frac{1}{2}$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}$$

78. (a) Let $\theta = \operatorname{arccot} 2$

$\cot \theta = 2$

$$\tan(\operatorname{arccot} 2) = \tan \theta = \frac{1}{2}$$

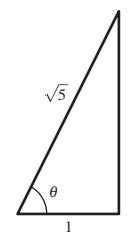


(b) Let $\theta = \operatorname{arcsec} \sqrt{5}$

$\sec \theta = \sqrt{5}$

$\cos(\operatorname{arcsec} \sqrt{5}) = \cos \theta$

$$= \frac{1}{\sqrt{5}}$$



79. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

80. $y = \arctan(x^2 - 1)$

$$y' = \frac{2x}{1+(x^2-1)^2} = \frac{2x}{x^4-2x^2+2}$$

81. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

82. $y = \frac{1}{2} \arctan e^{2x}$

$$y' = \frac{1}{2} \left(\frac{1}{1+e^{4x}} \right) (2e^{2x}) = \frac{e^{2x}}{1+e^{4x}}$$

83. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$$

84. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \frac{x}{2}, \quad 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2-4}} - \frac{1}{(|x|/2)\sqrt{(x/2)^2-1}} = \frac{x}{\sqrt{x^2-4}} - \frac{4}{|x|\sqrt{x^2-4}} = \frac{x^2-4}{|x|\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{x}$$

85. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

86. Let $u = 5x$, $du = 5 dx$.

$$\int \frac{1}{3 + 25x^2} dx = \frac{1}{5} \int \frac{1}{(\sqrt{3})^2 + (5x)^2} (5) dx = \frac{1}{5\sqrt{3}} \arctan \frac{5x}{\sqrt{3}} + C$$

87. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

$$88. \int \frac{1}{16+x^2} dx = \frac{1}{4} \arctan \frac{x}{4} + C$$

89. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\begin{aligned} \int \frac{\arctan(x/2)}{4+x^2} dx &= \frac{1}{2} \int \left(\arctan \frac{x}{2} \right) \left(\frac{2}{4+x^2} \right) dx \\ &= \frac{1}{4} \left(\arctan \frac{x}{2} \right)^2 + C \end{aligned}$$

90. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}} dx$.

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \frac{1}{2} (\arcsin x)^2 + C$$

$$91. A = \int_0^1 \frac{4-x}{\sqrt{4-x^2}} dx$$

$$= 4 \int_0^1 \frac{1}{\sqrt{4-x^2}} dx + \frac{1}{2} \int_0^1 (4-x^2)^{-1/2} (-2x) dx$$

$$= \left[4 \arcsin\left(\frac{x}{2}\right) + \sqrt{4-x^2} \right]_0^1$$

$$= \left(4 \arcsin\left(\frac{1}{2}\right) + \sqrt{3} \right) - 2$$

$$= \frac{2\pi}{3} + \sqrt{3} - 2 \approx 1.8264$$

$$92. \int_0^4 \frac{x}{16+x^2} dx = \frac{1}{2} \ln(16+x^2) \Big|_0^4$$

$$= \frac{1}{2} \ln 32 - \frac{1}{2} \ln 16$$

$$= \frac{1}{2} \ln 2$$

$$93. \int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}} dt$$

$$\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right).$$

$$94. y = 2x - \cosh \sqrt{x}$$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

95. $y = x \tanh^{-1} 2x$

$$y' = x \left(\frac{2}{1-4x^2} \right) + \tanh^{-1} 2x = \frac{2x}{1-4x^2} + \tanh^{-1} 2x$$

96. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$

97. Let $u = x^3$, $du = 3x^2 dx$.

$$\int x^2 (\operatorname{sech} x^3)^2 dx = \frac{1}{3} \int (\operatorname{sech} x^3)^2 (3x^2) dx = \frac{1}{3} \tanh x^3 + C$$

Problem Solving for Chapter 5

1. $\tan \theta_1 = \frac{3}{x}$

$$\tan \theta_2 = \frac{6}{10-x}$$

Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10-x}\right)$$

$$f'(x) = \frac{1}{1 + \frac{9}{x^2}} \left(-\frac{3}{x^2}\right) + \frac{1}{1 + \frac{36}{(10-x)^2}} \left(-\frac{6}{(10-x)^2}\right) = 0$$

$$\frac{3}{x^2 + 9} = \frac{6}{(10-x)^2 + 36}$$

$$(10-x)^2 + 36 = 2(x^2 + 9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

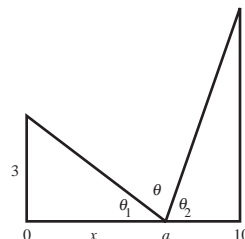
$$a = -10 + \sqrt{218} \approx 4.7648 \quad f(a) \approx 1.4153$$

$$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263 \quad \text{or} \quad 98.9^\circ$$

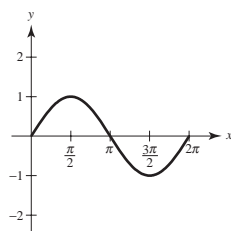
Endpoints: $a = 0$: $\theta \approx 1.0304$

$$a = 10$$
: $\theta \approx 1.2793$

Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.

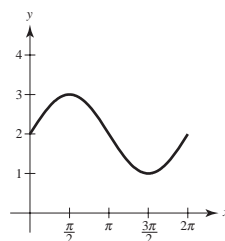


2. (a)



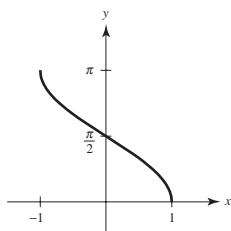
$$\int_0^{\pi} \sin x \, dx = -\int_{\pi}^{2\pi} \sin x \, dx \Rightarrow \int_0^{2\pi} \sin x \, dx = 0$$

(b)



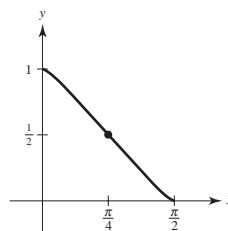
$$\int_0^{2\pi} (\sin x + 2) \, dx = 2(2\pi) = 4\pi$$

(c)



$$\int_{-1}^1 \arccos x \, dx = 2\left(\frac{\pi}{2}\right) = \pi$$

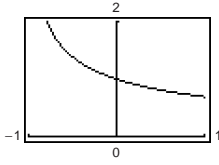
(d)



$$y = \frac{1}{1 + (\tan x)^2} \text{ symmetric with respect to point } \left(\frac{\pi}{4}, \frac{1}{2}\right).$$

$$\int_0^{\pi/2} \frac{1}{1 + (\tan x)^2} \, dx = \frac{\pi}{2} \left(\frac{1}{2}\right) = \frac{\pi}{4}$$

3. (a) $f(x) = \frac{\ln(x+1)}{x}, \quad -1 \leq x \leq 1$



(b) $\lim_{x \rightarrow 0} f(x) = 1$

4. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(c) $f(x) = -1 = \sin(\ln x) \Rightarrow \ln x = \frac{3\pi}{2} + 2k\pi$

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

(e) $f'(x) = \frac{1}{x} \cos(\ln x)$

$f'(x) = 0 \Rightarrow \cos(\ln x) = 0 \Rightarrow \ln x = \frac{\pi}{2} + k\pi \Rightarrow$

$x = e^{\pi/2}$ on $[1, 10]$

$f(e^{\pi/2}) = 1$
 $f(1) = 0$
 $f(10) \approx 0.7440$ } Maximum is 1 at $x = e^{\pi/2} \approx 4.8105$

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \dots$ we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \dots$ we have $f(x) = -1$.

That is, as $x \rightarrow 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus, $\lim_{x \rightarrow 0^+} \sin(\ln x)$ does not exist. You can verify this by graphing $f(x)$ on small intervals close to the origin.

(c) Let $g(x) = \ln x, g'(x) = 1/x$, and $g'(1) = 1$. From the definition of derivative

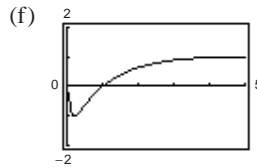
$$g'(1) = \lim_{x \rightarrow 0} \frac{g(1+x) - g(1)}{x} = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$$

Thus, $\lim_{x \rightarrow 0} f(x) = 1$.

(b) $f(x) = 1 = \sin(\ln x) \Rightarrow \ln x = \frac{\pi}{2} + 2k\pi$

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.



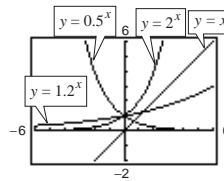
$\lim_{x \rightarrow 0^+} f(x)$ seems to be $-\frac{1}{2}$. (This is incorrect.)

5. $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$. $y = 2^x$ does not intersect $y = x$. Suppose $y = x$ is tangent to $y = a^x$ at (x, y) .

$a^x = x \Rightarrow a = x^{1/x}$

$y' = a^x \ln a = 1 \Rightarrow x \ln x^{1/x} = 1 \Rightarrow \ln x = 1 \Rightarrow x = e, a = e^{1/e}$

For $0 < a \leq e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.



6. (a) $\frac{\text{Area sector}}{\text{Area circle}} = \frac{t}{2\pi} \Rightarrow \text{Area sector} = \frac{t}{2\pi}(\pi) = \frac{t}{2}$

(b) $\text{Area } AOP = \frac{1}{2}(\text{base})(\text{height}) - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$

$A(t) = \frac{1}{2} \cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$

$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1} \sinh t = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sinh^2 t = \frac{1}{2}[\cosh^2 t - \sinh^2 t] = \frac{1}{2}$

$A(t) = \frac{1}{2}t + C$

But, $A(0) = C = 0 \Rightarrow C = 0$ Thus, $A(t) = \frac{1}{2}t$ or $t = 2A(t)$.

7. (a) $y = f(x) = \arcsin x$

$$\sin y = x$$

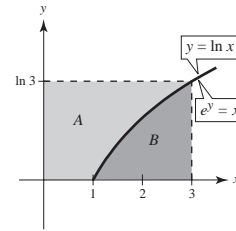
$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = -\cos y \Big|_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

$$\begin{aligned} \text{(b) } \int_{1/2}^{\sqrt{2}/2} \arcsin x \, dx &= \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B \\ &= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12} = \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346 \end{aligned}$$

$$\begin{aligned} \text{(c) Area } A &= \int_0^{\ln 3} e^y \, dy \\ &= e^y \Big|_0^{\ln 3} = 3 - 1 = 2 \end{aligned}$$

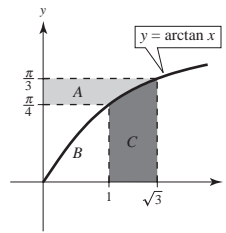
$$\text{Area } B = \int_1^3 \ln x \, dx = 3(\ln 3) - A = 3 \ln 3 - 2 = \ln 27 - 2 \approx 1.2958$$



$$\text{(d) } \tan y = x$$

$$\begin{aligned} \text{Area } A &= \int_{\pi/4}^{\pi/3} \tan y \, dy \\ &= -\ln|\cos y| \Big|_{\pi/4}^{\pi/3} \\ &= -\ln \frac{1}{2} + \ln \frac{\sqrt{2}}{2} = \ln \sqrt{2} = \frac{1}{2} \ln 2 \end{aligned}$$

$$\begin{aligned} \text{Area } C &= \int_1^{\sqrt{3}} \arctan x \, dx = \left(\frac{\pi}{3}\right)(\sqrt{3}) - \frac{1}{2} \ln 2 - \left(\frac{\pi}{4}\right)(1) \\ &= \frac{\pi}{12}(4\sqrt{3} - 3) - \frac{1}{2} \ln 2 \approx 0.6818 \end{aligned}$$



8. $y = \ln x$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$$y = \frac{1}{a}x + b - 1 \quad \text{Tangent line}$$

If $x = 0$, $c = b - 1$. Thus, $b - c = b - (b - 1) = 1$.

9. $y = e^x$

$$y' = e^x$$

$$y - b = e^a(x - a)$$

$$y = e^ax - ae^a + b \quad \text{Tangent line}$$

$$\text{If } y = 0: e^ax = ae^a - b$$

$$bx = ab - b \quad (b = e^a)$$

$$x = a - 1$$

$$c = a - 1$$

Thus, $a - c = a - (a - 1) = 1$.

10. Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$,
 $dx = (2u - 2) du$.

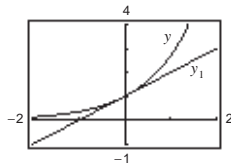
$$\begin{aligned} \text{Area} &= \int_1^4 \frac{1}{\sqrt{x} + x} dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)} du \\ &= \int_2^3 \frac{2(u - 1)}{u^2 - u} du \\ &= \int_2^3 \frac{2}{u} du = \left[2 \ln|u| \right]_2^3 \\ &= 2 \ln 3 - 2 \ln 2 = 2 \ln\left(\frac{3}{2}\right) \\ &\approx 0.8109 \end{aligned}$$

11. Let $u = \tan x$, $du = \sec^2 x dx$.

$$\begin{aligned} \text{Area} &= \int_0^{\pi/4} \frac{1}{\sin^2 x + 4 \cos^2 x} dx = \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4} dx \\ &= \int_0^1 \frac{du}{u^2 + 4} \\ &= \left[\frac{1}{2} \arctan\left(\frac{u}{2}\right) \right]_0^1 \\ &= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \end{aligned}$$

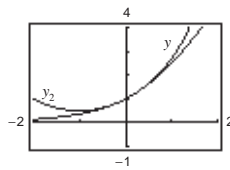
12. (a) (i) $y = e^x$

$$y_1 = 1 + x$$



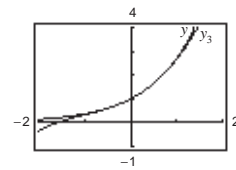
- (ii) $y = e^x$

$$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$$



- (iii) $y = e^x$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

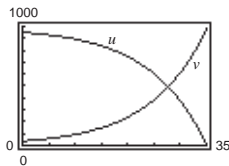


- (b) n^{th} term is $x^n/n!$ in polynomial: $y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

- (c) Conjecture: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

13. (a) $u = 985.93 - \left(985.93 - \frac{(120,000)(0.095)}{12}\right)\left(1 + \frac{0.095}{12}\right)^{12t}$

$$v = \left(985.93 - \frac{(120,000)(0.095)}{12}\right)\left(1 + \frac{0.095}{12}\right)^{12t}$$



- (b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

- (c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \implies \frac{du}{dt} = -\frac{dv}{dt}$$

$$u'(15) = -v'(15) = -14.06.$$

- (d) $t = 12.7$ years

Again, the larger part goes for interest.