CHAPTER 7
Applications of Integration

C H A P T E R 7
Applications of Integration

Section 7.1 Area of a Region Between Two Curves

1. $A = \int_0^6 [0 - (x^2 - 6x)]\,dx = -\int_0^6 (x^2 - 6x)\,dx$

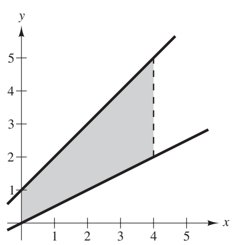
3. $A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)]\,dx$

$= \int_0^3 (-2x^2 + 6x)\,dx$

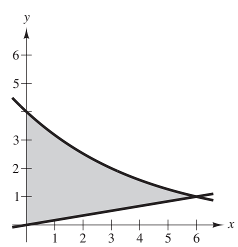
5. $A = 2\int_{-1}^0 3(x^3 - x)\,dx = 6\int_{-1}^0 (x^3 - x)\,dx$

or $-6\int_0^1 (x^3 - x)\,dx$

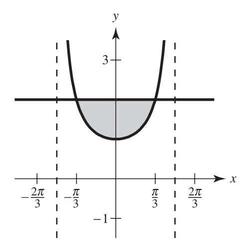
7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$



9. $\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6}\right] dx$



11. $\int_{-\pi/3}^{\pi/3} (2 - \sec x)\,dx$



13. (a)

$x = 4 - y^2$

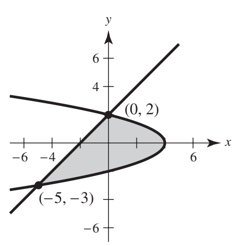
$x = y - 2$

$4 - y^2 = y - 2$

$y^2 + y - 6 = 0$

$(y + 3)(y - 2) = 0$

Intersection points: (0, 2) and (−5, −3)



$A = \int_{-5}^0 [(x + 2) + \sqrt{4 - x}\,dx + \int_0^4 2\sqrt{4 - x}\,dx = \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$

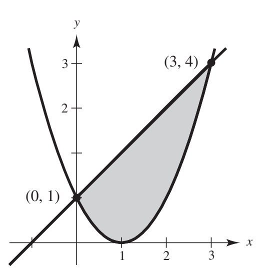
(b) $A = \int_{-3}^2 [(4 - y^2) - (y - 2)]\,dy = \frac{125}{6}$

15. $f(x) = x + 1$

$g(x) = (x - 1)^2$

$A \approx 4$

Matches (d)

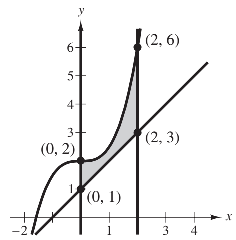


17. $A = \int_0^2 \left[\left(\frac{1}{2}x^3 + 2\right) - (x + 1)\right] dx$

$= \int_0^2 \left(\frac{1}{2}x^3 - x + 1\right) dx$

$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x\right]_0^2$

$= \left(\frac{16}{8} - \frac{4}{2} + 2\right) - 0 = 2$



19. The points of intersection are given by:

$x^2 - 4x = 0$

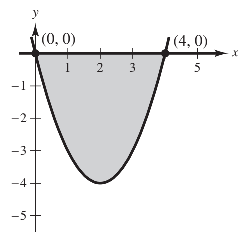
$x(x - 4) = 0$ when $x = 0, 4$

$A = \int_0^4 [g(x) - f(x)]\, dx$

$= -\int_0^4 (x^2 - 4x)\, dx$

$= -\left[\frac{x^3}{3} - 2x^2\right]_0^4$

$= \frac{32}{3}$



21. The points of intersection are given by:

$x^2 + 2x + 1 = 3x + 3$

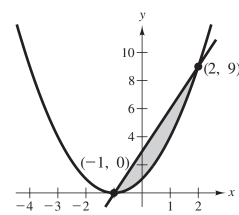
$(x - 2)(x + 1) = 0$ when $x = -1, 2$

$A = \int_{-1}^2 [g(x) - f(x)]\, dx$

$= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)]\, dx$

$= \int_{-1}^2 (2 + x - x^2)\, dx$

$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$



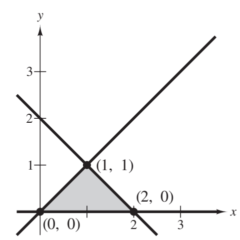
23. The points of intersection are given by:

$x = 2 - x$ and $x = 0$ and $2 - x = 0$

$x = 1$ $\quad x = 0$ $\quad x = 2$

$A = \int_0^1 [(2 - y) - (y)]\, dy = \left[2y - y^2\right]_0^1 = 1$

Note that if we integrate with respect to x, we need two integrals. Also, note that the region is a triangle.



25. The points of intersection are given by:

$\sqrt{3x} + 1 = x + 1$

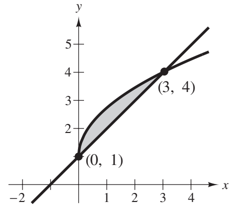
$\sqrt{3x} = x$ when $x = 0, 3$

$A = \int_0^3 [f(x) - g(x)]\, dx$

$= \int_0^3 \left[\left(\sqrt{3x} + 1\right) - (x + 1)\right] dx$

$= \int_0^3 [(3x)^{1/2} - x]\, dx$

$= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2}$



27. The points of intersection are given by:

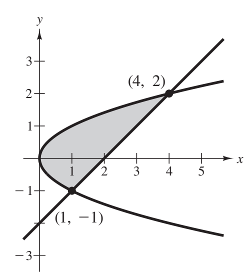
$y^2 = y + 2$

$(y - 2)(y + 1) = 0$ when $y = -1, 2$

$A = \int_{-1}^2 [g(y) - f(y)]\, dy$

$= \int_{-1}^2 [(y + 2) - y^2]\, dy$

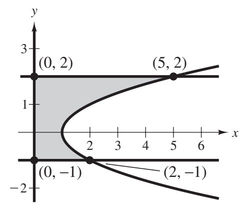
$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$



29. $A = \int_{-1}^{2} [f(y) - g(y)]\, dy$

$= \int_{-1}^{2} [(y^2 + 1) - 0]\, dy$

$= \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6$



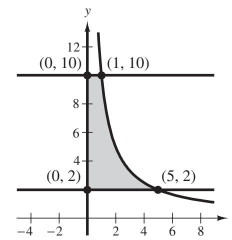
31. $y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$

$A = \int_{2}^{10} \frac{10}{y}\, dy$

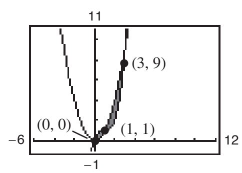
$= \left[10 \ln y\right]_{2}^{10}$

$= 10(\ln 10 - \ln 2)$

$= 10 \ln 5 \approx 16.0944$



33. (a)



(b) The points of intersection are given by:

$x^3 - 3x^2 + 3x = x^2$

$x(x - 1)(x - 3) = 0$ when $x = 0, 1, 3$

$A = \int_{0}^{1} [f(x) - g(x)]\, dx + \int_{1}^{3} [g(x) - f(x)]\, dx$

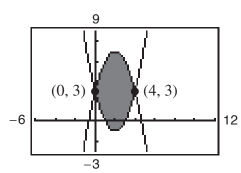
$= \int_{0}^{1} [(x^3 - 3x^2 + 3x) - x^2]\, dx + \int_{1}^{3} [x^2 - (x^3 - 3x^2 + 3x)]\, dx$

$= \int_{0}^{1} (x^3 - 4x^2 + 3x)\, dx + \int_{1}^{3} (-x^3 + 4x^2 - 3x)\, dx$

$= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_{0}^{1} + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_{1}^{3} = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$

(c) Numerical approximation:
$0.417 + 2.667 \approx 3.083$

35. (a)



(b) The points of intersection are given by:

$x^2 - 4x + 3 = 3 + 4x - x^2$

$2x(x - 4) = 0$ when $x = 0, 4$

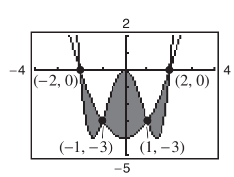
$A = \int_{0}^{4} [(3 + 4x - x^2) - (x^2 - 4x + 3)]\, dx$

$= \int_{0}^{4} (-2x^2 + 8x)\, dx$

$= \left[-\frac{2x^3}{3} + 4x^2\right]_{0}^{4} = \frac{64}{3}$

(c) Numerical approximation: 21.333

37. (a) $f(x) = x^4 - 4x^2,\ \ g(x) = x^2 - 4$



(b) The points of intersection are given by:

$x^4 - 4x^2 = x^2 - 4$

$x^4 - 5x^2 + 4 = 0$

$(x^2 - 4)(x^2 - 1) = 0$ when $x = \pm 2, \pm 1$

By symmetry:

$A = 2\int_{0}^{1} [(x^4 - 4x^2) - (x^2 - 4)]\, dx + 2\int_{1}^{2} [(x^2 - 4) - (x^4 - 4x^2)]\, dx$

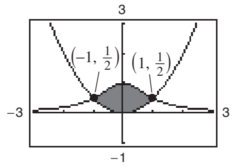
$= 2\int_{0}^{1} (x^4 - 5x^2 + 4)\, dx + 2\int_{1}^{2} (-x^4 + 5x^2 - 4)\, dx$

$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{0}^{1} + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{1}^{2}$

$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8$

(c) Numerical approximation:
$5.067 + 2.933 = 8.0$

39. (a)



(b) The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x = \pm 1$$

$$A = 2\int_0^1 [f(x) - g(x)]\,dx$$

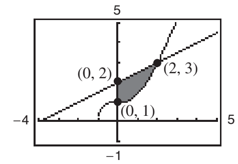
$$= 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2}\right]dx$$

$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

(c) Numerical approximation: 1.237

41. (a)



(b) and (c) $\sqrt{1+x^3} \le \frac{1}{2}x + 2$ on $[0, 2]$

You must use numerical integration because $y = \sqrt{1+x^3}$ does not have an elementary antiderivative.

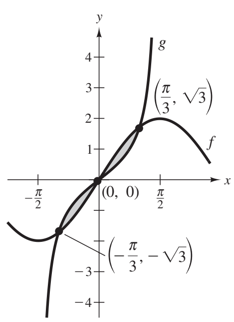
$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1+x^3}\right]dx \approx 1.759$$

43. $A = 2\int_0^{\pi/3} [f(x) - g(x)]\,dx$

$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx$$

$$= 2\Big[-2\cos x + \ln|\cos x|\Big]_0^{\pi/3}$$

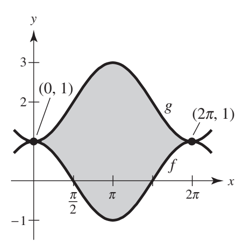
$$= 2(1 - \ln 2) \approx 0.614$$



45. $A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\,dx$

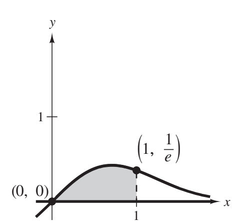
$$= 2\int_0^{2\pi} (1 - \cos x)\,dx$$

$$= 2\Big[x - \sin x\Big]_0^{2\pi} = 4\pi \approx 12.566$$

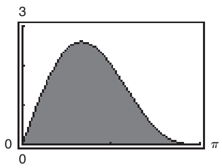


47. $A = \int_0^1 [xe^{-x^2} - 0]\,dx$

$$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$$



49. (a)



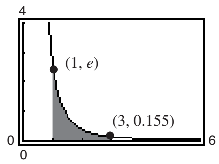
(b) $A = \int_0^{\pi} (2\sin x + \sin 2x)\, dx$

$= \left[-2\cos x - \frac{1}{2}\cos 2x\right]_0^{\pi}$

$= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) = 4$

(c) Numerical approximation: 4.0

51. (a)



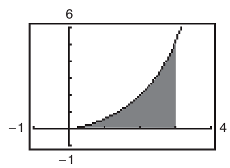
(b) $A = \int_1^3 \frac{1}{x^2} e^{1/x}\, dx$

$= \left[-e^{-1/x}\right]_1^3$

$= e - e^{1/3}$

(c) Numerical approximation: 1.323

53. (a)



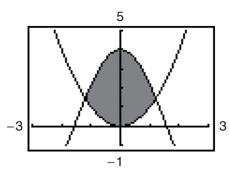
(b) The integral

$A = \int_0^3 \sqrt{\frac{x^3}{4-x}}\, dx$

does not have an elementary antiderivative.

(c) $A \approx 4.7721$

55. (a)

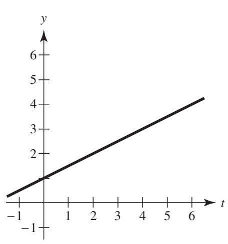


(b) The intersection points are difficult to determine by hand.

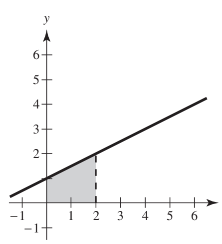
(c) Area $= \int_{-c}^{c} [4\cos x - x^2]\, dx \approx 6.3043$ where $c \approx 1.201538$.

57. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

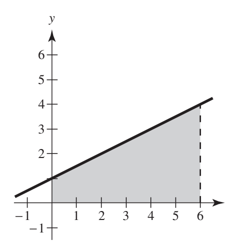
(a) $F(0) = 0$



(b) $F(2) = \frac{2^2}{4} + 2 = 3$

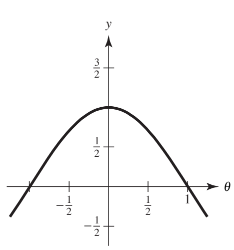


(c) $F(6) = \frac{6^2}{4} + 6 = 15$

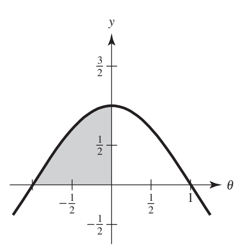


59. $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\, d\theta = \left[\frac{2}{\pi}\sin\frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2} + \frac{2}{\pi}$

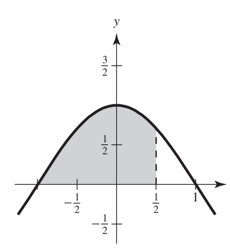
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$



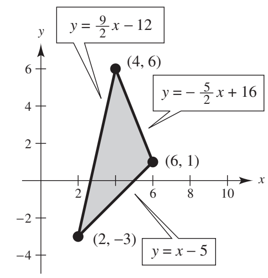
(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$



61. $A = \int_2^4 \left[\left(\frac{9}{2}x - 12\right) - (x - 5)\right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16\right) - (x - 5)\right] dx$

$= \int_2^4 \left(\frac{7}{2}x - 7\right) dx + \int_4^6 \left(-\frac{7}{2}x + 21\right) dx$

$= \left[\frac{7}{4}x^2 - 7x\right]_2^4 + \left[-\frac{7}{4}x^2 + 21x\right]_4^6 = 7 + 7 = 14$

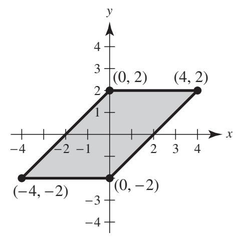


63. Left boundary line: $y = x + 2 \iff x = y - 2$

Right boundary line: $y = x - 2 \iff x = y + 2$

$A = \int_{-2}^{2} [(y + 2) - (y - 2)]\, dy$

$= \int_{-2}^{2} 4\, dy = 4y\Big]_{-2}^{2} = 8 - (-8) = 16$



65. Answers will vary. If you let $\Delta x = 6$ and $n = 10$, $b - a = 10(6) = 60$.

(a) Area $\approx \frac{60}{2(10)}[0 + 2(14) + 2(14) + 2(12) + 2(12) + 2(15) + 2(20) + 2(23) + 2(25) + 2(26) + 0]$

$= 3[322] = 966$ sq ft

(b) Area $\approx \frac{60}{3(10)}[0 + 4(14) + 2(14) + 4(12) + 2(12) + 4(15) + 2(20) + 4(23) + 2(25) + 4(26) + 0]$

$= 2[502] = 1004$ sq ft

67. $f(x) = x^3$

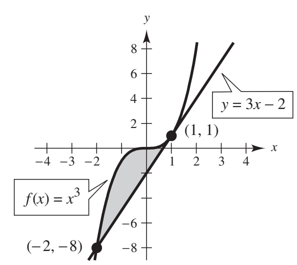
$f'(x) = 3x^2$

At $(1, 1)$, $f'(1) = 3$.

Tangent line: $y - 1 = 3(x - 1)$ or $y = 3x - 2$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$A = \int_{-2}^{1} [x^3 - (3x - 2)]\, dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \frac{27}{4}$



69. $f(x) = \frac{1}{x^2 + 1}$

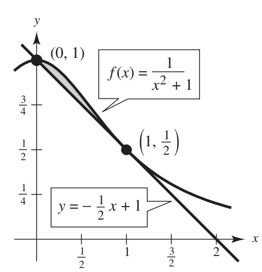
$f'(x) = -\frac{2x}{(x^2 + 1)^2}$

At $\left(1, \frac{1}{2}\right)$, $f'(1) = -\frac{1}{2}$.

Tangent line: $y - \frac{1}{2} = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x + 1$

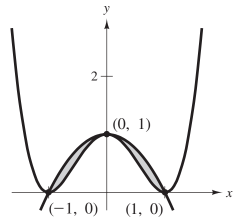
The tangent line intersects $f(x) = \frac{1}{x^2 + 1}$ at $x = 0$.

$A = \int_0^1 \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1\right)\right] dx = \left[\arctan x + \frac{x^2}{4} - x\right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$



71. $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

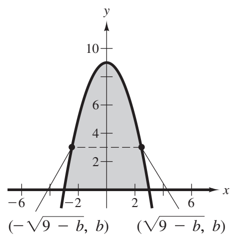
$$A = \int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\,dx$$
$$= \int_{-1}^{1} (x^2 - x^4)\,dx$$
$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$$



You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.

73. Offer 2 is better because the accumulated salary (area under the curve) is larger.

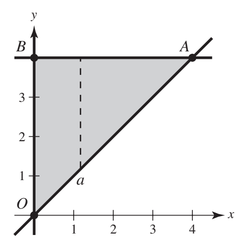
75. $$A = \int_{-3}^{3} (9 - x^2)\,dx = 36$$
$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b]\,dx = 18$$
$$\int_{0}^{\sqrt{9-b}} [(9 - b) - x^2]\,dx = 9$$
$$\left[(9 - b)x - \frac{x^3}{3}\right]_0^{\sqrt{9-b}} = 9$$
$$\frac{2}{3}(9 - b)^{3/2} = 9$$
$$(9 - b)^{3/2} = \frac{27}{2}$$
$$9 - b = \frac{9}{\sqrt[3]{4}}$$
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



77. Area of triangle OAB is $\frac{1}{2}(4)(4) = 8$.

$$4 = \int_0^a (4 - x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^a = 4a - \frac{a^2}{2}$$
$$a^2 - 8a + 8 = 0$$
$$a = 4 \pm 2\sqrt{2}$$

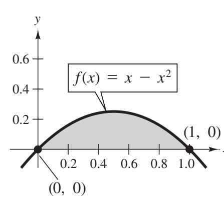
Since $0 < a < 4$, select $a = 4 - 2\sqrt{2} \approx 1.172$.



79. $$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (x_i - x_i^2)\,\Delta x$$

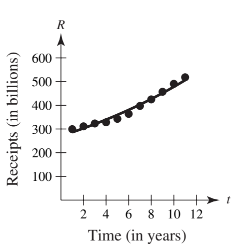
where $x_i = \dfrac{i}{n}$ and $\Delta x = \dfrac{1}{n}$ is the same as

$$\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

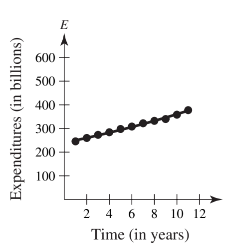


81. $$\int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)]\,dt = \int_0^5 0.13t\,dt = \left[\frac{0.13t^2}{2}\right]_0^5 = \$1.625 \text{ billion}$$

83. (a) $y_1 = (270.3151)(1.0586)^t = 270.3151e^{0.05695t}$



(b) $y_2 = (239.9704)(1.0416)^t = 239.9704e^{0.04074t}$



(c) Surplus $= \displaystyle\int_{12}^{17} (y_1 - y_2)\,dt \approx 926.4$ billion dollars

(Answers will vary.)

(d) No, $y_1 > y_2$ forever because $1.0586 > 1.0416$.
No, these models are not accurate for the future.
According to news, $E > R$ eventually.

85. 5%: $P_1 = 893{,}000e^{(0.05)t}$

$3\frac{1}{2}\%$: $P_2 = 893{,}000e^{(0.035)t}$

Difference in profits over 5 years: $\int_0^5 [893{,}000e^{0.05t} - 893{,}000e^{0.035t}]\,dt = 893{,}000\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5$

$$\approx 893{,}000[(25.6805 - 34.0356) - (20 - 28.5714)]$$

$$\approx 893{,}000(0.2163) \approx \$193{,}156$$

Note: Using a graphing utility, you obtain $193,183.

87. The curves intersect at the point where the slope of y_2 equals that of y_1, 1.

$$y_2 = 0.08x^2 + k \implies y'_2 = 0.16x = 1 \implies x = \frac{1}{0.16} = 6.25$$

(a) The value of k is given by

$$y_1 = y_2$$
$$6.25 = (0.08)(6.25)^2 + k$$
$$k = 3.125.$$

(b) $\text{Area} = 2\int_0^{6.25} (y_2 - y_1)\,dx$

$$= 2\int_0^{6.25} (0.08x^2 + 3.125 - x)\,dx$$

$$= 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25}$$

$$= 2(6.510417) \approx 13.02083$$

89. (a) $A \approx 6.031 - 2\left[\pi\left(\frac{1}{16}\right)^2\right] - 2\left[\pi\left(\frac{1}{8}\right)^2\right] \approx 5.908$

(b) $V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$

(c) $5000V \approx 5000(11.816) = 59{,}082$ pounds

91. True

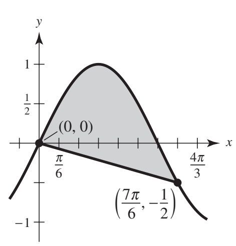
93. Line: $y = \frac{-3}{7\pi}x$

$$A = \int_0^{7\pi/6} \left[\sin x + \frac{3x}{7\pi}\right] dx$$

$$= \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6}$$

$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1$$

$$\approx 2.7823$$



95. We want to find c such that:

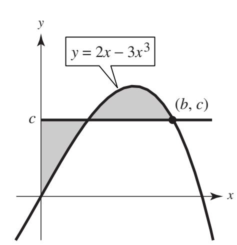
$$\int_0^b [(2x - 3x^3) - c]\,dx = 0$$

$$\left[x^2 - \frac{3}{4}x^4 - cx\right]_0^b = 0$$

$$b^2 - \frac{3}{4}b^4 - cb = 0$$

But, $c = 2b - 3b^3$ because (b, c) is on the graph.

$$b^2 - \frac{3}{4}b^4 - (2b - 3b^3)b = 0$$
$$4 - 3b^2 - 8 + 12b^2 = 0$$
$$9b^2 = 4$$
$$b = \frac{2}{3}$$
$$c = \frac{4}{9}$$



Section 7.2 Volume: The Disk Method

1. $V = \pi\int_0^1 (-x+1)^2\,dx = \pi\int_0^1 (x^2-2x+1)\,dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}$

3. $V = \pi\int_1^4 \left(\sqrt{x}\right)^2\,dx = \pi\int_1^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

5. $V = \pi\int_0^1 \left[(x^2)^2 - (x^3)^2\right]dx = \pi\int_0^1 (x^4 - x^6)\,dx = \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{2\pi}{35}$

7. $y = x^2 \Rightarrow x = \sqrt{y}$

$V = \pi\int_0^4 \left(\sqrt{y}\right)^2\,dy = \pi\int_0^4 y\,dy$

$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$

9. $y = x^{2/3} \Rightarrow x = y^{3/2}$

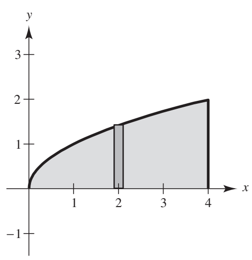
$V = \pi\int_0^1 (y^{3/2})^2\,dy = \pi\int_0^1 y^3\,dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$

11. $y = \sqrt{x},\ y = 0,\ x = 4$

(a) $R(x) = \sqrt{x},\ r(x) = 0$

$V = \pi\int_0^4 \left(\sqrt{x}\right)^2\,dx$

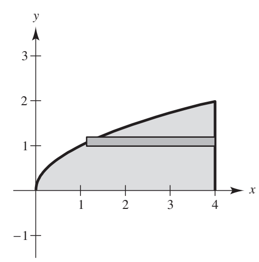
$= \pi\int_0^4 x\,dx = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$



(b) $R(y) = 4,\ r(y) = y^2$

$V = \pi\int_0^2 (16 - y^4)\,dy$

$= \pi\left[16y - \frac{1}{5}y^5\right]_0^2 = \frac{128\pi}{5}$

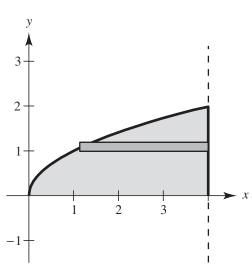


(c) $R(y) = 4 - y^2,\ r(y) = 0$

$V = \pi\int_0^2 (4 - y^2)^2\,dy$

$= \pi\int_0^2 (16 - 8y^2 + y^4)\,dy$

$= \pi\left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{256\pi}{15}$

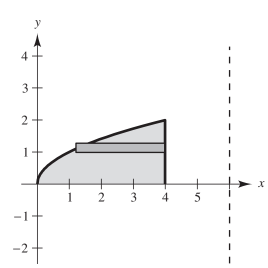


(d) $R(y) = 6 - y^2,\ r(y) = 2$

$V = \pi\int_0^2 \left[(6 - y^2)^2 - 4\right]dy$

$= \pi\int_0^2 (32 - 12y^2 + y^4)\,dy$

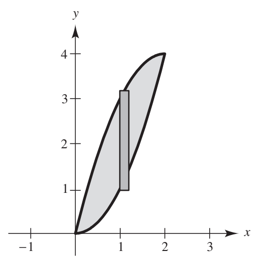
$= \pi\left[32y - 4y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{192\pi}{5}$



13. $y = x^2,\ y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

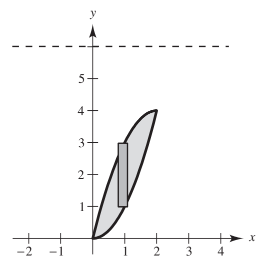
(a) $R(x) = 4x - x^2,\ r(x) = x^2$

$$V = \pi\int_0^2 [(4x - x^2)^2 - x^4]\,dx$$
$$= \pi\int_0^2 (16x^2 - 8x^3)\,dx$$
$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$



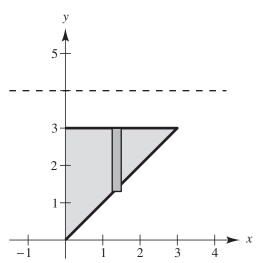
(b) $R(x) = 6 - x^2,\ r(x) = 6 - (4x - x^2)$

$$V = \pi\int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2]\,dx$$
$$= 8\pi\int_0^2 (x^3 - 5x^2 + 6x)\,dx$$
$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$



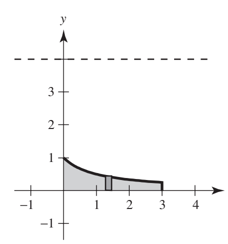
15. $R(x) = 4 - x,\ r(x) = 1$

$$V = \pi\int_0^3 [(4 - x)^2 - (1)^2]\,dx$$
$$= \pi\int_0^3 (x^2 - 8x + 15)\,dx$$
$$= \pi\left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3$$
$$= 18\pi$$



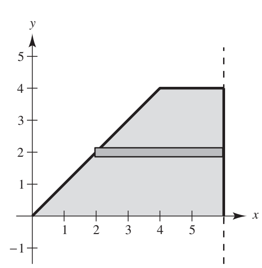
17. $R(x) = 4,\ r(x) = 4 - \dfrac{1}{1 + x}$

$$V = \pi\int_0^3 \left[4^2 - \left(4 - \frac{1}{1 + x}\right)^2\right]dx$$
$$= \pi\int_0^3 \left[\frac{8}{1 + x} - \frac{1}{(1 + x)^2}\right]dx$$
$$= \pi\left[8\ln(1 + x) + \frac{1}{1 + x}\right]_0^3$$
$$= \pi\left[8\ln 4 + \frac{1}{4} - 1\right]$$
$$= \left(8\ln 4 - \frac{3}{4}\right)\pi$$
$$\approx 32.485$$



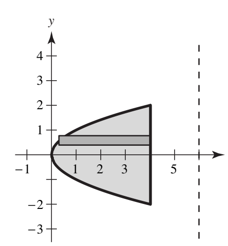
19. $R(y) = 6 - y,\ r(y) = 0$

$$V = \pi\int_0^4 (6 - y)^2\,dy$$
$$= \pi\int_0^4 (y^2 - 12y + 36)\,dy$$
$$= \pi\left[\frac{y^3}{3} - 6y^2 + 36y\right]_0^4$$
$$= \frac{208\pi}{3}$$



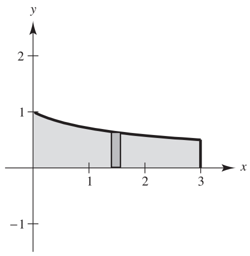
21. $R(y) = 6 - y^2,\ r(y) = 2$

$$V = \pi\int_{-2}^2 [(6 - y^2)^2 - (2)^2]\,dy$$
$$= 2\pi\int_0^2 (y^4 - 12y^2 + 32)\,dy$$
$$= 2\pi\left[\frac{y^5}{5} - 4y^3 + 32y\right]_0^2$$
$$= \frac{384\pi}{5}$$



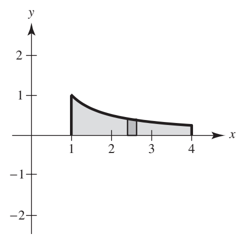
23. $R(x) = \dfrac{1}{\sqrt{x+1}},\ r(x) = 0$

$$V = \pi\int_0^3 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx$$
$$= \pi\int_0^3 \frac{1}{x+1}\,dx$$
$$= \Big[\pi \ln|x+1|\Big]_0^3$$
$$= \pi \ln 4 \approx 4.355$$



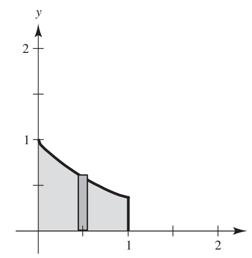
25. $R(x) = \dfrac{1}{x},\ r(x) = 0$

$$V = \pi\int_1^4 \left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left[-\frac{1}{x}\right]_1^4$$
$$= \frac{3\pi}{4}$$



27. $R(x) = e^{-x},\ r(x) = 0$

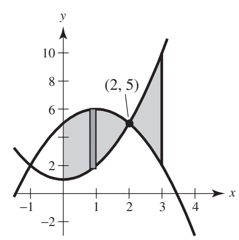
$$V = \pi\int_0^1 (e^{-x})^2\,dx$$
$$= \pi\int_0^1 e^{-2x}\,dx$$
$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$
$$= \frac{\pi}{2}(1 - e^{-2}) \approx 1.358$$



29.
$$x^2 + 1 = -x^2 + 2x + 5$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

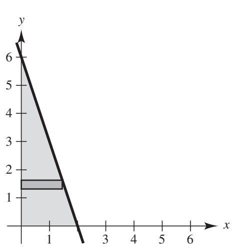
The curves intersect at $(-1, 2)$ and $(2, 5)$.

$$V = \pi\int_0^2 [(5 + 2x - x^2)^2 - (x^2+1)^2)]\,dx + \pi\int_2^3 [(x^2+1)^2 - (5 + 2x - x^2)^2]\,dx$$
$$= \pi\int_0^2 (-4x^3 - 8x^2 + 20x + 24)\,dx + \pi\int_2^3 (4x^3 + 8x^2 - 20x - 24)\,dx$$
$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$
$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$



31. $y = 6 - 3x \Longrightarrow x = \dfrac{1}{3}(6 - y)$

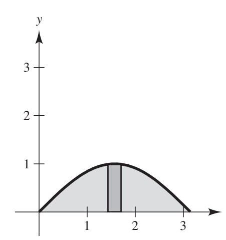
$$V = \pi\int_0^6 \left[\frac{1}{3}(6-y)\right]^2 dy$$
$$= \frac{\pi}{9}\int_0^6 [36 - 12y + y^2]\,dy$$
$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$
$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$
$$= 8\pi = \frac{1}{3}\pi r^2 h, \quad \text{Volume of cone}$$



33. $V = \pi\displaystyle\int_0^\pi (\sin x)^2\,dx$

$$= \pi\int_0^\pi \frac{1 - \cos 2x}{2}\,dx$$
$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$
$$= \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

Numerical approximation: 4.9348



35. $V = \pi\int_1^2 (e^{x-1})^2\,dx$

$= \pi\int_1^2 e^{2x-2}\,dx$

$= \frac{\pi}{2}e^{2x-2}\Big]_1^2$

$= \frac{\pi}{2}(e^2 - 1)$

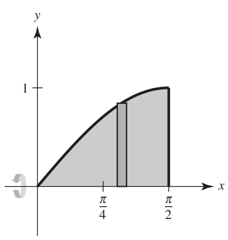
Numerical approximation:
10.0359

37. $V = \pi\int_0^2 [e^{-x^2}]^2\,dx \approx 1.9686$

39. $V = \pi\int_0^5 [2\arctan(0.2x)]^2\,dx$

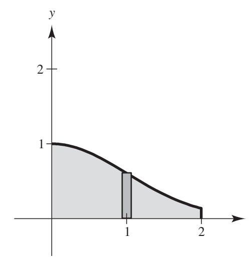
≈ 15.4115

41. $\pi\int_0^{\pi/2} \sin^2 x\,dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi/2$ about the x-axis.

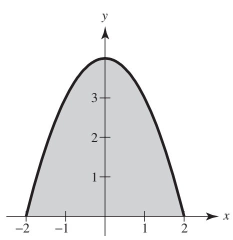


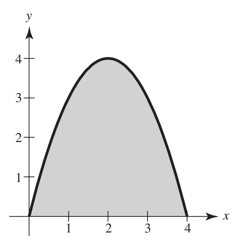
43. $A \approx 3$

Matches (a)



45.





The volumes are the same because the solid has been translated horizontally. $(4x - x^2 = 4 - (x-2)^2)$

47. $R(x) = \frac{1}{2}x,\ r(x) = 0$

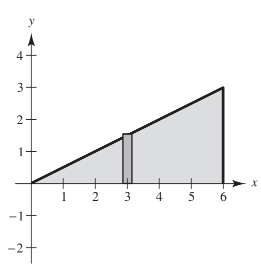
$V = \pi\int_0^6 \frac{1}{4}x^2\,dx$

$= \left[\frac{\pi}{12}x^3\right]_0^6 = 18\pi$

Note: $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(3^2)6$

$= 18\pi$



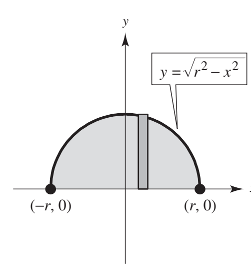
49. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$V = \pi\int_{-r}^{r} (r^2 - x^2)\,dx$

$= 2\pi\int_0^r (r^2 - x^2)\,dx$

$= 2\pi\left[r^2x - \frac{1}{3}x^3\right]_0^r$

$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$



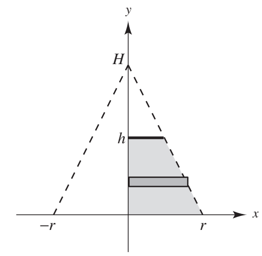
51. $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right),\ R(y) = r\left(1 - \frac{y}{H}\right),\ r(y) = 0$

$$V = \pi\int_0^h \left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right) dy$$

$$= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h$$

$$= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right)$$

$$= \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$$



53. $V = \pi\int_0^2 \left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\, dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}$

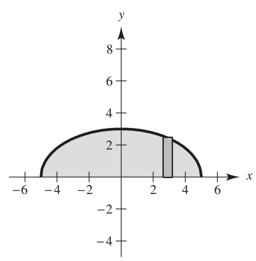
55. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2},\ r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^{5}(25 - x^2)\, dx$$

$$= \frac{18\pi}{25}\int_0^5 (25 - x^2)\, dx$$

$$= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5$$

$$= 60\pi$$

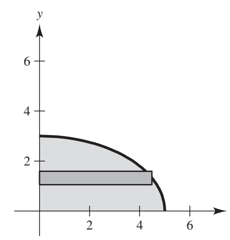


(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2},\ r(y) = 0,\ x \geq 0$

$$V = \frac{25\pi}{9}\int_0^3 (9 - y^2)\, dy$$

$$= \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3$$

$$= 50\pi$$



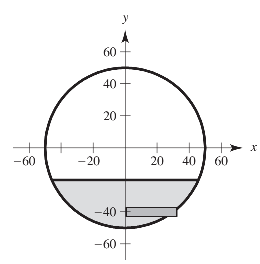
57. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500{,}000\pi}{3}\ \text{ft}^3$

Volume of water in the tank:

$$\pi\int_{-50}^{y_0} \left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0} (2500 - y^2)\, dy$$

$$= \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0}$$

$$= \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$



When the tank is one-fourth of its capacity:

$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$

$$y_0^3 - 7500y_0 - 125{,}000 = 0$$

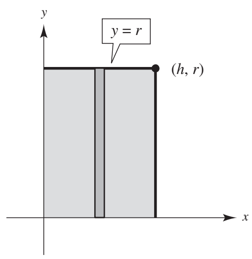
$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

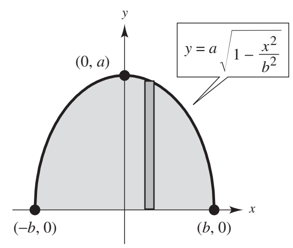
59. (a) $\pi\int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



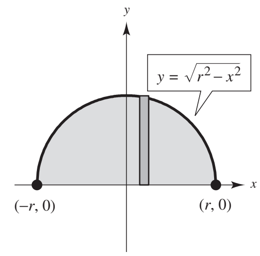
(b) $\pi\int_{-b}^{b}\left(a\sqrt{1-\frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



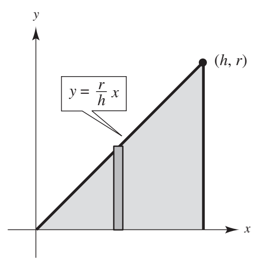
(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



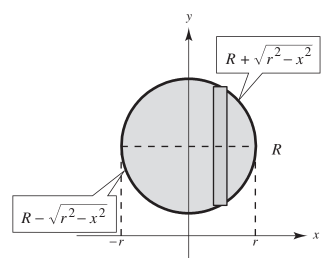
(d) $\pi\int_0^h\left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.

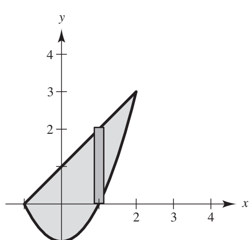


(e) $\pi\int_{-r}^{r}\left[\left(R+\sqrt{r^2-x^2}\right)^2-\left(R-\sqrt{r^2-x^2}\right)^2\right]dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



61.



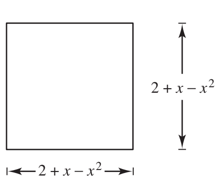
Base of cross section $= (x+1)-(x^2-1) = 2+x-x^2$

(a) $A(x) = b^2 = (2+x-x^2)^2$

$$= 4+4x-3x^2-2x^3+x^4$$

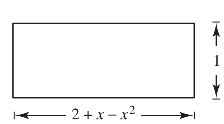
$$V = \int_{-1}^{2}(4+4x-3x^2-2x^3+x^4)\,dx$$

$$= \left[4x+2x^2-x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

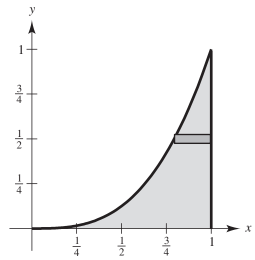


(b) $A(x) = bh = (2+x-x^2)1$

$$V = \int_{-1}^{2}(2+x-x^2)\,dx = \left[2x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



63.



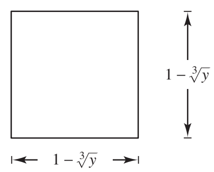
Base of cross section $= 1 - \sqrt[3]{y}$

(a) $A(y) = b^2 = \left(1 - \sqrt[3]{y}\right)^2$

$$V = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy$$

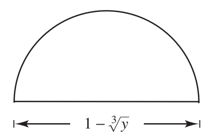
$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3})\, dy$$

$$= \left[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3}\right]_0^1 = \frac{1}{10}$$



(b) $A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1 - \sqrt[3]{y}}{2}\right)^2 = \frac{1}{8}\pi\left(1 - \sqrt[3]{y}\right)^2$

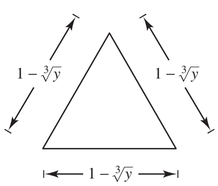
$$V = \frac{1}{8}\pi\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{8}\left(\frac{1}{10}\right) = \frac{\pi}{80}$$



(c) $A(y) = \frac{1}{2}bh = \frac{1}{2}(1 - \sqrt[3]{y})\left(\frac{\sqrt{3}}{2}\right)\left(1 - \sqrt[3]{y}\right)$

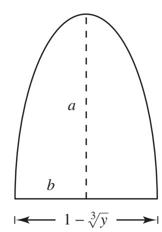
$$= \frac{\sqrt{3}}{4}\left(1 - \sqrt[3]{y}\right)^2$$

$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\sqrt{3}}{4}\left(\frac{1}{10}\right) = \frac{\sqrt{3}}{40}$$



(d) $A(y) = \frac{1}{2}\pi ab = \frac{\pi}{2}(2)\left(1 - \sqrt[3]{y}\right)\frac{1 - \sqrt[3]{y}}{2} = \frac{\pi}{2}\left(1 - \sqrt[3]{y}\right)^2$

$$V = \frac{\pi}{2}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{2}\left(\frac{1}{10}\right) = \frac{\pi}{20}$$



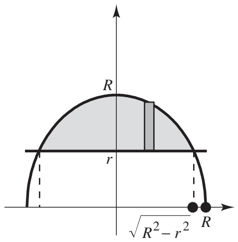
65. $V = \pi\int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$

$$= 2\pi\int_0^{\sqrt{R^2 - r^2}} (R^2 - r^2 - x^2)\, dx$$

$$= 2\pi\left[(R^2 - r^2)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2 - r^2}}$$

$$= 2\pi\left[(R^2 - r^2)^{3/2} - \frac{(R^2 - r^2)^{3/2}}{3}\right]$$

$$= \frac{4}{3}\pi(R^2 - r^2)^{3/2}$$



67. $V = \pi\int_0^1 y^2\, dy = \pi\frac{y^3}{3}\bigg]_0^1 = \frac{\pi}{3}$

69. $V = \pi\int_0^1 (x^2 - x^4)\,dx$

$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= \pi\left[\frac{1}{3} - \frac{1}{5}\right]$

$= \frac{2\pi}{15}$

71. $V = \pi\int_0^1 (1 - y)\,dy$

$= \pi\left[y - \frac{y^2}{2}\right]_0^1$

$= \pi\left[1 - \frac{1}{2}\right]$

$= \frac{\pi}{2}$

73. $V = \pi\int_0^1 (y - y^2)\,dy$

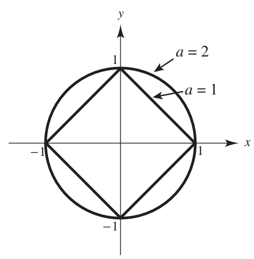
$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$

$= \pi\left[\frac{1}{2} - \frac{1}{3}\right]$

$= \frac{\pi}{6}$

75. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.



(b) $|y| = (1 - |x|^a)^{1/a}$

$A = 2\int_{-1}^{1} (1 - |x|^a)^{1/a}\,dx = 4\int_0^1 (1 - x^a)^{1/a}\,dx$

To approximate the volume of the solid, form n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.

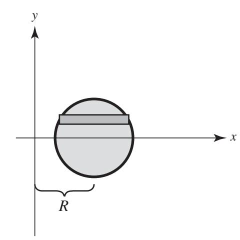
77. (a) $(x - R)^2 + y^2 = r^2$

$x = R \pm \sqrt{r^2 - y^2}$

$V = 2\pi\int_0^r \left(\left[R + \sqrt{r^2 - y^2}\right]^2 - \left[R - \sqrt{r^2 - y^2}\right]^2\right) dy$

$= 2\pi\int_0^r 4R\sqrt{r^2 - y^2}\,dy$

$= 8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy$



(b) $\int_0^r \sqrt{r^2 - y^2}\,dy$ is one-quarter of the area of a circle of radius r, $\frac{1}{4}\pi r^2$.

$V = 8\pi R\left(\frac{1}{4}\pi r^2\right) = 2\pi^2 r^2 R$

Section 7.3 Volume: The Shell Method

1. $p(x) = x,\ h(x) = x$

$V = 2\pi\int_0^2 x(x)\,dx$

$= \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$

3. $p(x) = x,\ h(x) = \sqrt{x}$

$V = 2\pi\int_0^4 x\sqrt{x}\,dx$

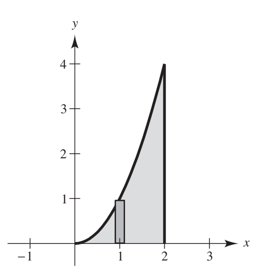
$= 2\pi\int_0^4 x^{3/2}\,dx$

$= \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$

5. $p(x) = x,\ h(x) = x^2$

$V = 2\pi\int_0^2 x^3\,dx$

$= \left[\frac{\pi}{2}x^4\right]_0^2 = 8\pi$

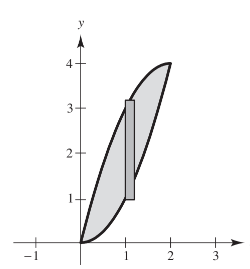


7. $p(x) = x,\ h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$V = 2\pi\int_0^2 x(4x - 2x^2)\,dx$

$= 4\pi\int_0^2 (2x^2 - x^3)\,dx$

$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$



9. $p(x) = x$

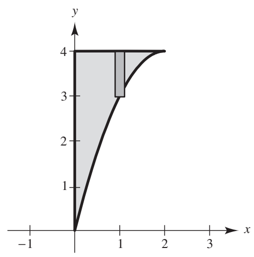
$h(x) = 4 - (4x - x^2)$

$= x^2 - 4x + 4$

$$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2$$

$$= \frac{8\pi}{3}$$



11. $p(x) = x,\ h(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

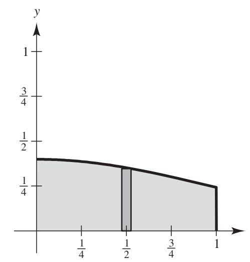
$$V = 2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx$$

$$= \sqrt{2\pi}\int_0^1 e^{-x^2/2}x\,dx$$

$$= \left[-\sqrt{2\pi}e^{-x^2/2}\right]_0^1$$

$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$

$$\approx 0.986$$



13. $p(y) = y,\ h(y) = 2 - y$

$$V = 2\pi\int_0^2 y(2 - y)\,dy$$

$$= 2\pi\int_0^2 (2y - y^2)\,dy$$

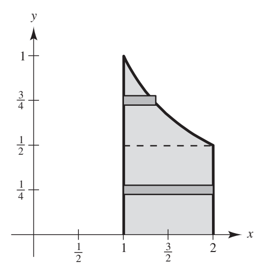
$$= 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \le y < \dfrac{1}{2}$.

$p(y) = y$ and $h(y) = \dfrac{1}{y} - 1$ if $\dfrac{1}{2} \le y \le 1$.

$$V = 2\pi\int_0^{1/2} y\,dy + 2\pi\int_{1/2}^1 (1 - y)\,dy$$

$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



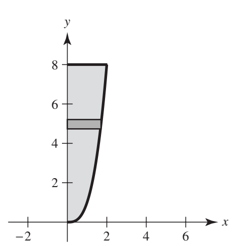
17. $p(y) = y,\ h(y) = \sqrt[3]{y}$

$$V = 2\pi\int_0^8 y\sqrt[3]{y}\,dy$$

$$= 2\pi\int_0^8 y^{4/3}\,dy$$

$$= \left[2\pi\left(\frac{3}{7}\right)y^{7/3}\right]_0^8$$

$$= \frac{6\pi}{7}(2^7) = \frac{768\pi}{7}$$



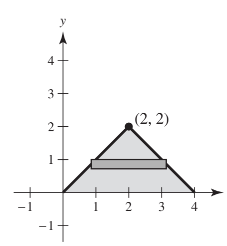
19. $p(y) = y,\ h(y) = (4 - y) - (y) = 4 - 2y$

$$V = 2\pi\int_0^2 y(4 - 2y)\,dy$$

$$= 2\pi\int_0^2 (4y - 2y^2)\,dy$$

$$= 2\pi\left[2y^2 - \frac{2}{3}y^3\right]_0^2$$

$$= 2\pi\left[8 - \frac{16}{3}\right] = \frac{16\pi}{3}$$

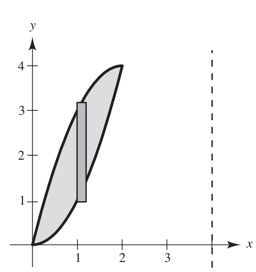


21. $p(x) = 4 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 (4 - x)(4x - 2x^2)\,dx$$

$$= 2\pi(2)\int_0^2 (x^3 - 6x^2 + 8x)\,dx$$

$$= 4\pi\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 16\pi$$

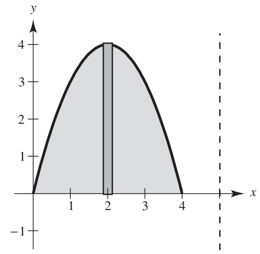


23. $p(x) = 5 - x,\ h(x) = 4x - x^2$

$$V = 2\pi\int_0^4 (5 - x)(4x - x^2)\,dx$$

$$= 2\pi\int_0^4 (x^3 - 9x^2 + 20x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 64\pi$$



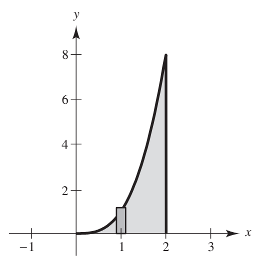
25. The shell method would be easier: $V = 2\pi\int_0^4 \left[4 - (y - 2)^2\right] y\,dy$ shells

Using the disk method: $V = \pi\int_0^4 \left[\left(2 + \sqrt{4 - x}\right)^2 - \left(2 - \sqrt{4 - x}\right)^2\right] dx$ $\left[\textbf{Note: } V = \frac{128\pi}{3}\right]$

27. (a) **Disk**

$R(x) = x^3,\ r(x) = 0$

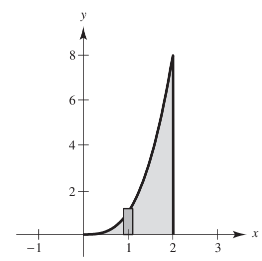
$$V = \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$



(b) **Shell**

$p(x) = x,\ h(x) = x^3$

$$V = 2\pi\int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$



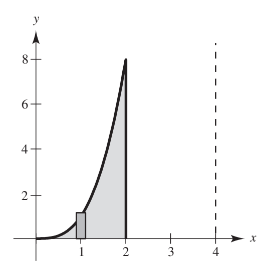
(c) **Shell**

$p(x) = 4 - x,\ h(x) = x^3$

$$V = 2\pi\int_0^2 (4 - x)x^3\,dx$$

$$= 2\pi\int_0^2 (4x^3 - x^4)\,dx$$

$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$



29. (a) **Shell**

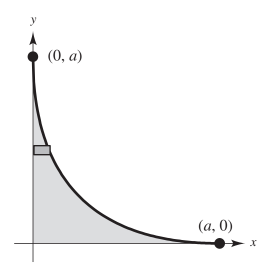
$p(y) = y,\ h(y) = (a^{1/2} - y^{1/2})^2$

$$V = 2\pi\int_0^a y(a - 2a^{1/2}y^{1/2} + y)\,dy$$

$$= 2\pi\int_0^a (ay - 2a^{1/2}y^{3/2} + y^2)\,dy$$

$$= 2\pi\left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi\left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right] = \frac{\pi a^3}{15}$$



(b) Same as part (a) by symmetry

—CONTINUED—

29. —CONTINUED—

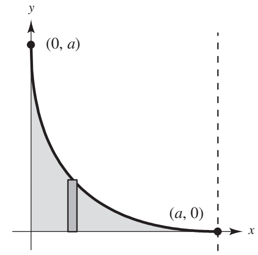
(c) **Shell**

$p(x) = a - x,\ h(x) = (a^{1/2} - x^{1/2})^2$

$$V = 2\pi\int_0^a (a - x)(a^{1/2} - x^{1/2})^2\, dx$$

$$= 2\pi\int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2)\, dx$$

$$= 2\pi\left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a = \frac{4\pi a^3}{15}$$



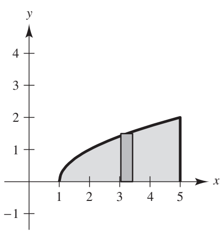
31. Answers will vary. (a) The rectangles would be vertical. (b) The rectangles would be horizontal.

33. $\pi\int_1^5 (x - 1)\, dx = \pi\int_1^5 \left(\sqrt{x - 1}\right)^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x-axis by using the disk method.

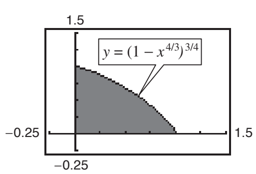
$$2\pi\int_0^2 y[5 - (y^2 + 1)]\, dy$$

represents this same volume by using the shell method.



Disk method

35. (a)

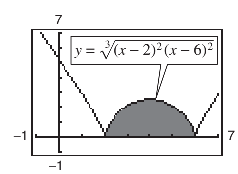


(b) $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$

$y = (1 - x^{4/3})^{3/4}$

$$V = 2\pi\int_0^1 x(1 - x^{4/3})^{3/4}\, dx \approx 1.5056$$

37. (a)

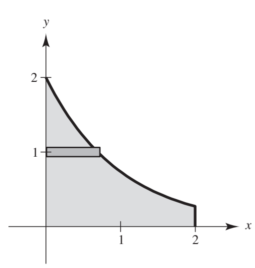


(b) $V = 2\pi\int_2^6 x\sqrt[3]{(x - 2)^2(x - 6)^2}\, dx \approx 187.249$

39. $y = 2e^{-x},\ y = 0,\ x = 0,\ x = 2$

Volume ≈ 7.5

Matches (d)



41. $p(x) = x,\ h(x) = 2 - \frac{1}{2}x^2$

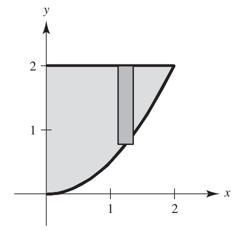
$$V = 2\pi\int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx = 2\pi\int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \quad \text{(total volume)}$$

Now find x_0 such that:

$$\pi = 2\pi\int_0^{x_0}\left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$



$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \qquad \text{(Quadratic Formula)}$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$, since the other root is too large.

Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

43. $V = 4\pi\int_{-1}^{1}(2 - x)\sqrt{1 - x^2}\, dx$

$$= 8\pi\int_{-1}^{1}\sqrt{1 - x^2}\, dx - 4\pi\int_{-1}^{1} x\sqrt{1 - x^2}\, dx$$

$$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi\int_{-1}^{1} x(1 - x^2)^{1/2}(-2)\, dx$$

$$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_{-1}^{1} = 4\pi^2$$

45. (a) $\frac{d}{dx}[\sin x - x\cos x + C] = \cos x + x\sin x - \cos x = x\sin x$

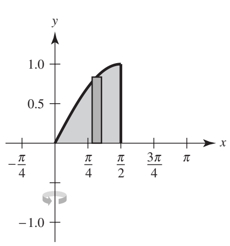
Hence, $\int x\sin x\, dx = \sin x - x\cos x + C.$

(b) (i) $p(x) = x, h(x) = \sin x$

$$V = 2\pi\int_0^{\pi/2} x\sin x\, dx$$

$$= 2\pi\left[\sin x - x\cos x\right]_0^{\pi/2}$$

$$= 2\pi[(1 - 0) - 0] = 2\pi$$



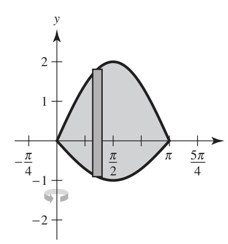
(ii) $p(x) = x, h(x) = 2\sin x - (-\sin x) = 3\sin x$

$$V = 2\pi\int_0^{\pi} x(3\sin x)\, dx$$

$$= 6\pi\int_0^{\pi} x\sin x\, dx$$

$$= 6\pi\left[\sin x - x\cos x\right]_0^{\pi}$$

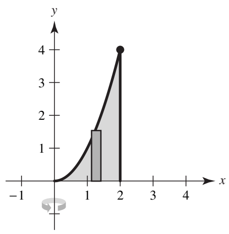
$$= 6\pi[\pi] = 6\pi^2$$



47. $2\pi\int_0^2 x^3\,dx = 2\pi\int_0^2 x(x^2)\,dx$

(a) Plane region bounded by $y = x^2$, $y = 0$, $x = 0$, $x = 2$

(b) Revolved about the y-axis

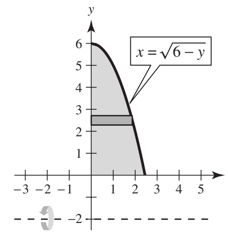


Other answers possible

49. $2\pi\int_0^6 (y + 2)\sqrt{6 - y}\,dy$

(a) Plane region bounded by $x = \sqrt{6 - y}$, $x = 0$, $y = 0$

(b) Revolved around line $y = -2$



Other answers possible

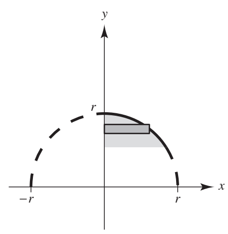
51. Disk Method

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

$$V = \pi\int_{r-h}^{r} (r^2 - y^2)\,dy$$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$$



53. (a) Area region $= \int_0^b [ab^n - ax^n]\,dx$

$$= \left[ab^nx - a\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= ab^{n+1} - a\frac{b^{n+1}}{n+1}$$

$$= ab^{n+1}\left(1 - \frac{1}{n+1}\right) = ab^{n+1}\left(\frac{n}{n+1}\right)$$

$$R_1(n) = \frac{ab^{n+1}[n/(n+1)]}{(ab^n)b} = \frac{n}{n+1}$$

(b) $\lim_{n\to\infty} R_1(n) = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\lim_{n\to\infty} (ab^n)b = \infty$

(c) **Disk Method:**

$$V = 2\pi\int_0^b x(ab^n - ax^n)\,dx$$

$$= 2\pi a\int_0^b (xb^n - x^{n+1})\,dx$$

$$= 2\pi a\left[\frac{b^n}{2}x^2 - \frac{x^{n+2}}{n+2}\right]_0^b$$

$$= 2\pi a\left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\frac{n}{n+2}\right)$$

$$R_2(n) = \frac{\pi ab^{n+2}[n/(n+2)]}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2}\right)$$

(d) $\lim_{n\to\infty} R_2(n) = \lim_{n\to\infty}\left(\frac{n}{n+2}\right) = 1$

$\lim_{n\to\infty} (\pi b^2)(ab^n) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = 1$.

55. (a) $V = 2\pi\int_0^4 xf(x)\,dx$

$$= \frac{2\pi(40)}{3(4)}[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0]$$

$$= \frac{20\pi}{3}[5800] \approx 121{,}475 \text{ cubic feet}$$

—CONTINUED—

55. —CONTINUED—

(b) Top line: $y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \implies y = -\frac{1}{2}x + 50$

Bottom line: $y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \implies y = -2x + 80$

$$V = 2\pi \int_0^{20} x\left(-\frac{1}{2}x + 50\right) dx + 2\pi \int_{20}^{40} x(-2x + 80)\, dx$$

$$= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x\right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x)\, dx$$

$$= 2\pi\left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40}$$

$$= 2\pi\left[\frac{26{,}000}{3}\right] + 2\pi\left[\frac{32{,}000}{3}\right]$$

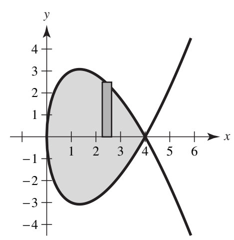
$\approx$ 121,475 cubic feet

(Note that Simpson's Rule is exact for this problem.)

57. $y^2 = x(4 - x)^2, \quad 0 \le x \le 4$

$y_1 = \sqrt{x(4 - x)^2} = (4 - x)\sqrt{x}$

$y_2 = -\sqrt{x(4 - x)^2} = -(4 - x)\sqrt{x}$



(a) $V = \pi \int_0^4 x(4 - x)^2\, dx$

$$= \pi \int_0^4 (x^3 - 8x^2 + 16x)\, dx$$

$$= \pi\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^4 = \frac{64\pi}{3}$$

(b) $V = 4\pi \int_0^4 x(4 - x)\sqrt{x}\, dx$

$$= 4\pi \int_0^4 (4x^{3/2} - x^{5/2})\, dx$$

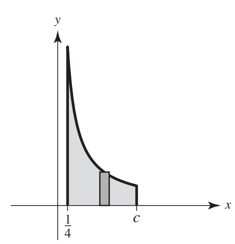
$$= 4\pi\left[\frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{2048\pi}{35}$$

(c) $V = 4\pi \int_0^4 (4 - x)(4 - x)\sqrt{x}\, dx$

$$= 4\pi \int_0^4 \left(16\sqrt{x} - 8x^{3/2} + x^{5/2}\right) dx$$

$$= 4\pi\left[\frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{8192\pi}{105}$$

59. $V_1 = \pi \int_{1/4}^{c} \frac{1}{x^2}\, dx = \pi\left[-\frac{1}{x}\right]_{1/4}^{c} = \pi\left[-\frac{1}{c} + 4\right] = \frac{4c - 1}{c}\pi$



$V_2 = 2\pi \int_{1/4}^{c} x\left(\frac{1}{x}\right) dx = 2\pi x\Big]_{1/4}^{c} = 2\pi\left(c - \frac{1}{4}\right)$

$V_1 = V_2 \implies \frac{4c - 1}{c}\pi = 2\pi\left(c - \frac{1}{4}\right)$

$$4c - 1 = 2c\left(c - \frac{1}{4}\right)$$

$$4c^2 - 9c + 2 = 0$$

$$(4c - 1)(c - 2) = 0$$

$$c = 2 \quad \left(c = \frac{1}{4} \text{ yields no volume.}\right)$$

Section 7.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2}$

$= 13$

(b) $y = \dfrac{12}{5}x$

$$y' = \frac{12}{5}$$

$$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2}\,dx$$

$$= \left[\frac{13}{5}x\right]_0^5 = 13$$

3. $y = \dfrac{2}{3}x^{3/2} + 1$

$$y' = x^{1/2}, \quad 0 \le x \le 1$$

$$s = \int_0^1 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$$

5. $y = \dfrac{3}{2}x^{2/3}$

$$y' = \frac{1}{x^{1/3}}, \quad 1 \le x \le 8$$

$$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx$$

$$= \int_1^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx$$

$$= \frac{3}{2}\int_1^8 \sqrt{x^{2/3}+1}\left(\frac{2}{3x^{1/3}}\right)dx$$

$$= \frac{3}{2}\left[\frac{2}{3}(x^{2/3}+1)^{3/2}\right]_1^8$$

$$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$$

7. $y = \dfrac{x^5}{10} + \dfrac{1}{6x^3}$

$$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, \quad 1 \le x \le 2$$

$$s = \int_a^b \sqrt{1 + (y')^2}\,dx$$

$$= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2}\,dx$$

$$= \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)dx$$

$$= \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.2458$$

9. $y = \ln(\sin x), \quad \left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$

$$y' = \frac{1}{\sin x}\cos x = \cot x$$

$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$

$$s = \int_{\pi/4}^{3\pi/4} \csc x\,dx$$

$$= \Big[\ln\left|\csc x - \cot x\right|\Big]_{\pi/4}^{3\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right) \approx 1.763$$

11. $y = \dfrac{1}{2}(e^x + e^{-x})$

$$y' = \frac{1}{2}(e^x - e^{-x}), \quad [0, 2]$$

$$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, \quad [0, 2]$$

$$s = \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}\,dx$$

$$= \frac{1}{2}\int_0^2 (e^x + e^{-x})\,dx$$

$$= \frac{1}{2}\Big[e^x - e^{-x}\Big]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

13. $x = \dfrac{1}{3}(y^2+2)^{3/2}, \quad 0 \le y \le 4$

$$\frac{dx}{dy} = y(y^2+2)^{1/2}$$

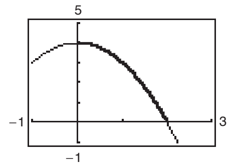
$$s = \int_0^4 \sqrt{1 + y^2(y^2+2)}\,dy$$

$$= \int_0^4 \sqrt{y^4 + 2y^2 + 1}\,dy$$

$$= \int_0^4 (y^2+1)\,dy$$

$$= \left[\frac{y^3}{3} + y\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

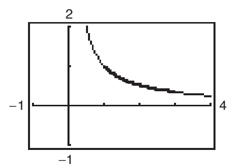
15. (a) $y = 4 - x^2, \quad 0 \le x \le 2$



(b) $$y' = -2x$$
$$1 + (y')^2 = 1 + 4x^2$$
$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx$$

(c) $L \approx 4.647$

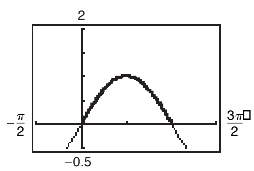
17. (a) $y = \dfrac{1}{x}, \quad 1 \le x \le 3$



(b) $$y' = -\frac{1}{x^2}$$
$$1 + (y')^2 = 1 + \frac{1}{x^4}$$
$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

19. (a) $y = \sin x, \quad 0 \le x \le \pi$



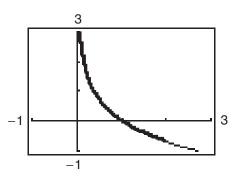
(b) $$y' = \cos x$$
$$1 + (y')^2 = 1 + \cos^2 x$$
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

21. (a) $x = e^{-y}, \quad 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $$y' = -\frac{1}{x}$$
$$1 + (y')^2 = 1 + \frac{1}{x^2}$$
$$L = \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

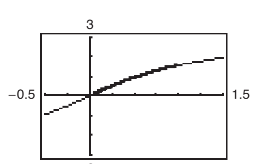
Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y}, \quad 0 \le y \le 2$

(b) $$\frac{dx}{dy} = -e^{-y}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$
$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

23. (a) $y = 2 \arctan x, \quad 0 \le x \le 1$



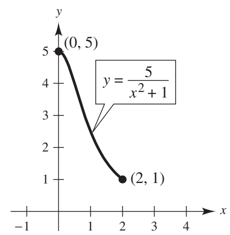
(b) $$y' = \frac{2}{1 + x^2}$$
$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\, dx$$

(c) $L \approx 1.871$

25. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2+1}\right)\right]^2}\, dx$

$s \approx 5$

Matches (b)



27. $y = x^3, \quad [0, 4]$

(a) $d = \sqrt{(4-0)^2 + (64-0)^2} \approx 64.125$

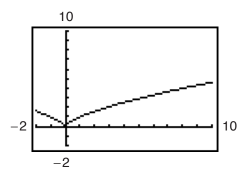
(b) $d = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} + \sqrt{(3-2)^2 + (27-8)^2} + \sqrt{(4-3)^2 + (64-27)^2}$

≈ 64.525

(c) $\displaystyle s = \int_0^4 \sqrt{1 + (3x^2)^2}\, dx = \int_0^4 \sqrt{1 + 9x^4}\, dx \approx 64.666$ (Simpson's Rule, $n = 10$)

(d) 64.672

29. (a) $f(x) = x^{2/3}$



(b) No, $f'(0)$ is not defined.

(c) $\displaystyle f'(x) = \frac{2}{3}x^{-1/3}$

$$1 + f'(x)^2 = 1 + \frac{4}{9x^{2/3}} = \frac{9x^{2/3}+4}{9x^{2/3}}$$

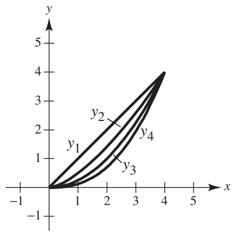
Divide $[-1, 8]$ into two intervals.

$$[-1, 0]: \; s_1 = \int_{-1}^0 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx$$
$$= \frac{-1}{3}\int_{-1}^0 \sqrt{9x^{2/3}+4}\,\frac{1}{x^{1/3}}\, dx, \quad (x0)$$
$$= -\frac{1}{18}\int_{-1}^0 (9x^{2/3}+4)^{1/2}\left(\frac{6}{x^{1/3}}\right) dx$$
$$= -\frac{1}{27}(9x^{2/3}+4)^{3/2}\Big]_{-1}^0$$
$$= -\frac{1}{27}(4^{3/2} - 13^{3/2})$$
$$= -\frac{1}{27}(8 - 13^{3/2}) \approx 1.4397$$

$$[0, 8]: \; s_2 = \int_0^8 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx$$
$$= \frac{1}{3}\int_0^8 \sqrt{9x^{2/3}+4}\,\frac{1}{x^{1/3}}\, dx, \quad (x \ge 0)$$
$$= \frac{1}{27}(9x^{2/3}+4)^{3/2}\Big]_0^8$$
$$= \frac{1}{27}(40^{3/2} - 4^{3/2})$$
$$= \frac{1}{27}(40^{3/2} - 8) \approx 9.0734$$

$$s_1 + s_2 = \frac{1}{27}[40^{3/2} - 8 - 8 + 13^{3/2}]$$
$$= \frac{1}{27}[40^{3/2} + 13^{3/2} - 16] \approx 10.5131$$

31. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1,\ L_1 = \int_0^4 \sqrt{2}\, dx \approx 5.657$

$y_2' = \frac{3}{4}x^{1/2},\ L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\, dx \approx 5.759$

$y_3' = \frac{1}{2}x,\ L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\, dx \approx 5.916$

$y_4' = \frac{5}{16}x^{3/2},\ L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\, dx \approx 6.063$

33. $y = \frac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0$, $y = \frac{2}{3}$. Thus, the fleeing object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3}\left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right] = \left(\frac{1}{2}\right)\frac{x-1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x+1}{2x^{1/2}}\, dx = \frac{1}{2}\int_0^1 (x^{1/2} + x^{-1/2})\, dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

35.

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

$$y' = \sinh\frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\, dx = 2\int_0^{20} \cosh\frac{x}{20}\, dx$$

$$= 2(20)\sinh\frac{x}{20}\Big]_0^{20} = 40\sinh(1) \approx 47.008 \text{ m}$$

37.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\, dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= \left[3\arcsin\frac{x}{3}\right]_0^2 = 3\left(\arcsin\frac{2}{3} - \arcsin 0\right)$$

$$= 3\arcsin\frac{2}{3} \approx 2.1892$$

39. $y = \frac{x^3}{3}$

$y' = x^2, \quad [0, 3]$

$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\, dx$$

$$= \frac{\pi}{6}\int_0^3 (1 + x^4)^{1/2}(4x^3)\, dx$$

$$= \left[\frac{\pi}{9}(1 + x^4)^{3/2}\right]_0^3$$

$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

41.

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \quad [1, 2]$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$

$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

43. $y = \sqrt[3]{x} + 2$

$$y' = \frac{1}{3x^{2/3}}, \quad [1, 8]$$

$$S = 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\,dx$$

$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\,dx$$

$$= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\,dx$$

$$= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

45. $y = \sin x$

$$y' = \cos x, \quad [0, \pi]$$

$$S = 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\,dx$$

$$\approx 14.4236$$

47. A rectifiable curve is one that has a finite arc length.

49. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

51.

$$y = \frac{hx}{r}$$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi \int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\,dx$$

$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

53.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}}\,dx$$

$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\,dx$$

$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$

$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 54.

55.

$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2})\,dx$$

$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in.}^2$$

Amount of glass needed: $V = \frac{\pi}{27}\left(\frac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in.}^3$

57. (a) We approximate the volume by summing six disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

$$V \approx \sum_{i=1}^{6} \pi r_i^2(3) = \sum_{i=1}^{6} \pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6} C_i^2$$

$$= \frac{3}{4\pi}\left[\left(\frac{50 + 65.5}{2}\right)^2 + \left(\frac{65.5 + 70}{2}\right)^2 + \left(\frac{70 + 66}{2}\right)^2 + \left(\frac{66 + 58}{2}\right)^2 + \left(\frac{58 + 51}{2}\right)^2 + \left(\frac{51 + 48}{2}\right)^2\right]$$

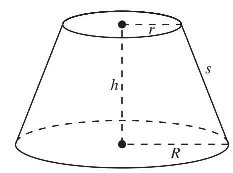
$$= \frac{3}{4\pi}[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2]$$

$$= \frac{3}{4\pi}[21813.625] = 5207.62 \text{ cubic inches}$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum:

$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$

$$= \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}.$$



Adding the six frustums together:

$$S \approx \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5 + 70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2} +$$

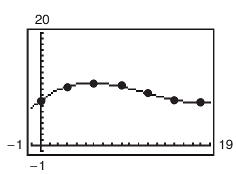
$$\left(\frac{70 + 66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66 + 58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2} +$$

$$\left(\frac{58 + 51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51 + 48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$

$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37$$

$$= 1168.64$$

(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$

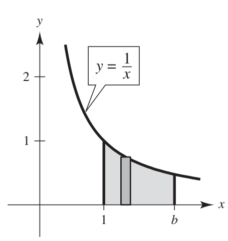


(d) $V = \int_0^{18} \pi r^2\,dy \approx 5275.9$ cubic inches

$$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\,dy$$

$$\approx 1179.5 \text{ square inches}$$

59. (a) $V = \pi\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$



(b) $S = 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$

$$= 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$$

$$= 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Since

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \Big[\ln x\Big]_1^b = \ln b$$

and $\lim_{b\to\infty} \ln b \to \infty$. Thus,

$$\lim_{b\to\infty} 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx = \infty.$$

61. Individual project

63. $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$

$$y = (4 - x^{2/3})^{3/2}, \quad 0 \le x \le 8$$

$$y' = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$

$$S = 2\pi\int_0^8 (4 - x^{2/3})^{3/2}\sqrt{\frac{4}{x^{2/3}}}\,dx$$

$$= 4\pi\int_0^8 \frac{(4 - x^{2/3})^{3/2}}{x^{1/3}}\,dx$$

$$= \left[-\frac{12\pi}{5}(4 - x^{2/3})^{5/2}\right]_0^8 = \frac{192\pi}{5}$$

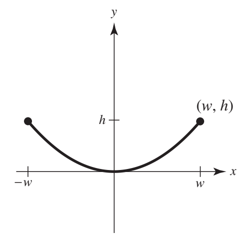
[Surface area of portion above the x-axis]

65. $y = kx^2$, $y' = 2kx$

$$1 + (y')^2 = 1 + 4k^2x^2$$

$$h = kw^2 \implies k = \frac{h}{w^2} \implies 1 + (y') = 1 + \frac{4h^2}{w^4}x^2$$

By symmetry, $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx.$



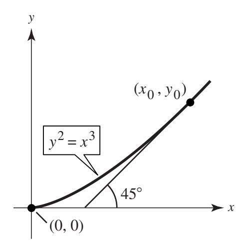
67. Let (x_0, y_0) be the point on the graph of $y^2 = x^3$ where the tangent line makes an angle of 45° with the x-axis.

$$y = x^{3/2}$$

$$y' = \tfrac{3}{2}x^{1/2} = 1$$

$$x_0 = \tfrac{4}{9}$$

$$L = \int_0^{4/9}\sqrt{1 + \tfrac{9}{4}x}\,dx = \tfrac{8}{27}\left(2\sqrt{2} - 1\right)$$



Section 7.5 Work

1. $W = Fd = (100)(10)$

$$= 1000 \text{ ft} \cdot \text{lb}$$

3. $W = Fd = (112)(4)$

$$= 448 \text{ joules (newton-meters)}$$

5. Work equals force times distance, $W = FD$.

7. Since the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

9. $F(x) = kx$

$$5 = k(4)$$

$$k = \frac{5}{4}$$

$$W = \int_0^7 \frac{5}{4}x\,dx = \left[\frac{5}{8}x^2\right]_0^7$$

$$= \frac{245}{8} \text{ in.} \cdot \text{lb}$$

$$= 30.625 \text{ in.} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$$

11. $F(x) = kx$

$$250 = k(30) \implies k = \frac{25}{3}$$

$$W = \int_{20}^{50} F(x)\,dx$$

$$= \int_{20}^{50} \frac{25}{3}x\,dx = \left.\frac{25x^2}{6}\right]_{20}^{50}$$

$$= 8750 \text{ n} \cdot \text{cm}$$

$$= 87.5 \text{ joules or Nm}$$

13. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$W = \int_0^{12} \frac{20}{9} x\,dx = \left[\frac{10}{9}x^2\right]_0^{12} = 160 \text{ in.} \cdot \text{lb} = \frac{40}{3} \text{ ft} \cdot \text{lb}$

15. $W = 18 = \int_0^{1/3} kx\,dx = \frac{kx^2}{2}\Big]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$W = \int_{1/3}^{7/12} 324x\,dx = 162x^2\Big]_{1/3}^{7/12} = 37.125 \text{ ft} \cdot \text{lbs}$

[**Note:** 4 inches $= \frac{1}{3}$ foot]

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$5 = \frac{k}{(4000)^2}$

$k = 80{,}000{,}000$

$F(x) = \frac{80{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{4100} \frac{80{,}000{,}000}{x^2}\,dx = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4100}$

$\approx 487.8 \text{ mi} \cdot \text{tons} \approx 5.15 \times 10^9 \text{ ft} \cdot \text{lb}$

(b) $W = \int_{4000}^{4300} \frac{80{,}000{,}000}{x^2}\,dx$

$\approx 1395.3 \text{ mi} \cdot \text{ton} \approx 1.47 \times 10^{10} \text{ ft} \cdot \text{ton}$

19. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$10 = \frac{k}{(4000)^2}$

$k = 160{,}000{,}000$

$F(x) = \frac{160{,}000{,}000}{x^2}$

(a) $W = \int_{4000}^{15{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333 \text{ mi} \cdot \text{ton}$

$\approx 2.93 \times 10^4 \text{ mi} \cdot \text{ton}$

$\approx 3.10 \times 10^{11} \text{ ft} \cdot \text{lb}$

(b) $W = \int_{4000}^{26{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154 \text{ mi} \cdot \text{ton}$

$\approx 3.38 \times 10^4 \text{ mi} \cdot \text{ton}$

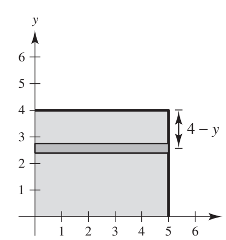
$\approx 3.57 \times 10^{11} \text{ ft} \cdot \text{lb}$

21. Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_2^4 = 2496 \text{ ft} \cdot \text{lb}$

(b) $W = \int_0^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_0^4 = 9984 \text{ ft} \cdot \text{lb}$



23. Volume of disk: $\pi(2)^2\,\Delta y = 4\pi\,\Delta y$

Weight of disk of water: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: $5 - y$

$W = \int_0^4 (5 - y)(9800)4\pi\,dy = 39{,}200\pi \int_0^4 (5 - y)\,dy$

$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$

$= 39{,}200\pi(12) = 470{,}400\pi$ newton–meters

25. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\,\Delta y$

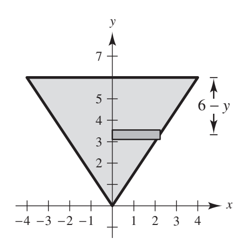
Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\,\Delta y$

Distance: $6 - y$

$W = \frac{4(62.4)\pi}{9}\int_0^6 (6 - y)y^2\,dy$

$= \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6$

$= 2995.2\pi \text{ ft} \cdot \text{lb}$



27. Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2 \Delta y$

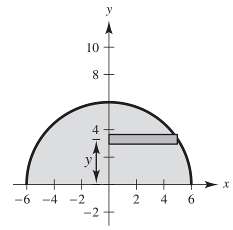
Weight of disk: $62.4\pi(36 - y^2)\,\Delta y$

Distance: y

$$W = 62.4\pi \int_0^6 y(36 - y^2)\,dy$$

$$= 62.4\pi \int_0^6 (36y - y^3)\,dy = 62.4\pi\left[18y^2 - \frac{1}{4}y^4\right]_0^6$$

$$= 20{,}217.6\pi \text{ ft} \cdot \text{lb}$$

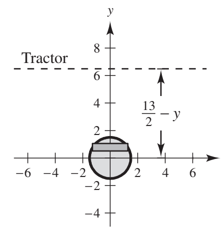


29. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$

Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

Distance: $\frac{13}{2} - y$

$$W = \int_{-1.5}^{1.5} 42(8)\sqrt{\frac{9}{4} - y^2}\left(\frac{13}{2} - y\right)dy = 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{\frac{9}{4} - y^2}\,y\,dy\right]$$



The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336\left(\frac{13}{2}\right)\pi\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) = 2457\pi \text{ ft} \cdot \text{lb}.$$

31. Weight of section of chain: $3\,\Delta y$

Distance: $15 - y$

$$W = 3\int_0^{15} (15 - y)\,dy$$

$$= \left[-\frac{3}{2}(15 - y)^2\right]_0^{15}$$

$$= 337.5 \text{ ft} \cdot \text{lb}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section: $3\,\Delta y$

Distance: $10 - y$

$$W_2 = 3\int_0^{10} (10 - y)\,dy$$

$$= \left[-\frac{3}{2}(10 - y)^2\right]_0^{10} = 150 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

35. Weight of section of chain: $3\,\Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5} (15 - 2y)\,dy = \left[-\frac{3}{4}(15 - 2y)^2\right]_0^{7.5}$$

$$= \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

37. Work to pull up the ball: $W_1 = 500(15) = 7500 \text{ ft} \cdot \text{lb}$

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1\,\Delta y$

Distance: $15 - x$

$$W_2 = \int_0^{15} (15 - x)\,dx = \left[-\frac{1}{2}(15 - x)^2\right]_0^{15}$$

$$= 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375$$

$$= 7987.5 \text{ ft} \cdot \text{lb}$$

39. $p = \frac{k}{V}$

$1000 = \frac{k}{2}$

$k = 2000$

$W = \int_2^3 \frac{2000}{V}\,dV$

$= \Big[2000 \ln|V|\Big]_2^3$

$= 2000 \ln\left(\frac{3}{2}\right) \approx 810.93$ ft · lb

41. $F(x) = \frac{k}{(2-x)^2}$

$W = \int_{-2}^{1} \frac{k}{(2-x)^2}\,dx$

$= \left[\frac{k}{2-x}\right]_{-2}^{1} = k\left(1 - \frac{1}{4}\right)$

$= \frac{3k}{4}$ (units of work)

43. $W = \int_0^5 1000[1.8 - \ln(x+1)]\,dx \approx 3249.44$ ft · lb

45. $W = \int_0^5 100x\sqrt{125 - x^3}\,dx \approx 10{,}330.3$ ft · lb

Section 7.6 Moments, Centers of Mass, and Centroids

1. $\bar{x} = \frac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\frac{6}{7}$

3. $\bar{x} = \frac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

5. (a) $\bar{x} = \frac{(7+5) + (8+5) + (12+5) + (15+5) + (18+5)}{5} = 17 = 12 + 5$

(b) $\bar{x} = \frac{12(-6-3) + 1(-4-3) + 6(-2-3) + 3(0-3) + 11(8-3)}{12 + 1 + 6 + 3 + 11} = \frac{-99}{33} = -3$

7. $50x = 75(L - x) = 75(10 - x)$

$50x = 750 - 75x$

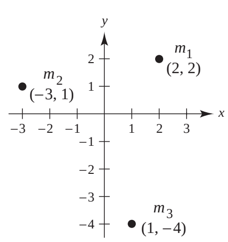
$125x = 750$

$x = 6$ feet

9. $\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$

$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$

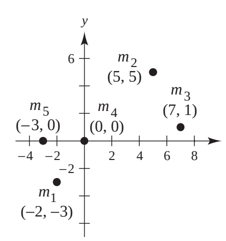
$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$



11. $\bar{x} = \frac{3(-2) + 4(5) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = \frac{5}{8}$

$\bar{y} = \frac{3(-3) + 4(5) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = \frac{13}{16}$

$(\bar{x}, \bar{y}) = \left(\frac{5}{8}, \frac{13}{16}\right)$



13. $m = \rho \int_0^4 \sqrt{x}\, dx = \left[\frac{2\rho}{3} x^{3/2}\right]_0^4 = \frac{16\rho}{3}$

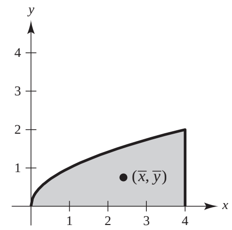
$M_x = \rho \int_0^4 \frac{\sqrt{x}}{2}\left(\sqrt{x}\right) dx = \left[\rho \frac{x^2}{4}\right]_0^4 = 4\rho$

$\bar{y} = \frac{M_x}{m} = 4\rho\left(\frac{3}{16\rho}\right) = \frac{3}{4}$

$M_y = \rho \int_0^4 x\sqrt{x}\, dx = \left[\rho \frac{2}{5} x^{5/2}\right]_0^4 = \frac{64\rho}{5}$

$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5}\left(\frac{3}{16\rho}\right) = \frac{12}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$



15. $m = \rho \int_0^1 (x^2 - x^3)\, dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$

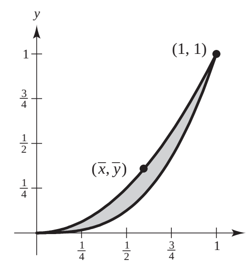
$M_x = \rho \int_0^1 \frac{(x^2 + x^3)}{2}(x^2 - x^3)\, dx = \frac{\rho}{2}\int_0^1 (x^4 - x^6)\, dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$

$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35}\left(\frac{12}{\rho}\right) = \frac{12}{35}$

$M_y = \rho \int_0^1 x(x^2 - x^3)\, dx = \rho \int_0^1 (x^3 - x^4)\, dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$

$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20}\left(\frac{12}{\rho}\right) = \frac{3}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$



17. $m = \rho \int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\, dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$

$M_x = \rho \int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2}\right][(-x^2 + 4x + 2) - (x + 2)]\, dx$

$= \frac{\rho}{2}\int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x)\, dx = \frac{\rho}{2}\int_0^3 (x^4 - 8x^3 + 11x^2 + 12x)\, dx$

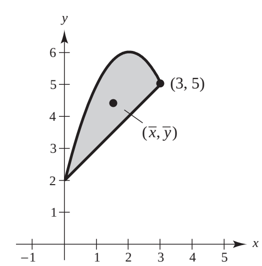
$= \frac{\rho}{2}\left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2\right]_0^3 = \frac{99\rho}{5}$

$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{22}{5}$

$M_y = \rho \int_0^3 x[(-x^2 + 4x - 2) - (x + 2)]\, dx = \rho \int_0^3 (-x^3 + 3x^2)\, dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$

$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5}\right)$



19. $m = \rho \int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$

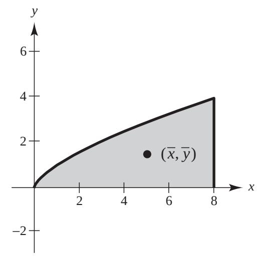
$M_x = \rho \int_0^8 \frac{x^{2/3}}{2}(x^{2/3})\,dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$

$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$

$M_y = \rho \int_0^8 x(x^{2/3})\,dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$

$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$

$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$



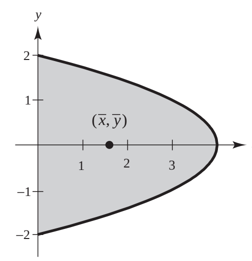
21. $m = 2\rho \int_0^2 (4 - y^2)\,dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$

$M_y = 2\rho \int_0^2 \left(\frac{4 - y^2}{2}\right)(4 - y^2)\,dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$

By symmetry, M_x and $\bar{y} = 0$.

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$



23. $m = \rho \int_0^3 [(2y - y^2) - (-y)]\,dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$

$M_y = \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2}[(2y - y^2) - (-y)]\,dy = \frac{\rho}{2}\int_0^3 (y - y^2)(3y - y^2)\,dy$

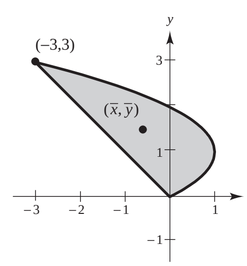
$= \frac{\rho}{2}\int_0^3 (y^4 - 4y^3 + 3y^2)\,dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$

$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$

$M_x = \rho \int_0^3 y[(2y - y^2) - (-y)]\,dy = \rho \int_0^3 (3y^2 - y^3)\,dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$



25. $A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

$M_x = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$

$M_y = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}$

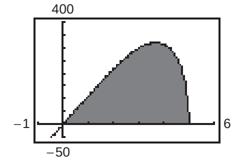
27. $A = \int_0^3 (2x + 4)\,dx = \left[x^2 + 4x\right]_0^3 = 9 + 12 = 21$

$M_x = \frac{1}{2}\int_0^3 (2x + 4)^2\,dx = \int_0^3 (2x^2 + 8x + 8)\,dx$

$= \left[\frac{2x^3}{3} + 4x^2 + 8x\right]_0^3 = 18 + 36 + 24 = 78$

$M_y = \int_0^3 (2x^2 + 4x)\,dx = \left[\frac{2x^3}{3} + 2x^2\right]_0^3 = 18 + 18 = 36$

29. $m = \rho \int_0^5 10x\sqrt{125 - x^3}\,dx \approx 1033.0\rho$



$$M_x = \rho \int_0^5 \left(\frac{10x\sqrt{125 - x^3}}{2}\right)\left(10x\sqrt{125 - x^3}\right) dx$$

$$= 50\rho \int_0^5 x^2(125 - x^3)\,dx = \frac{3{,}124{,}375\rho}{24} \approx 130{,}208\rho$$

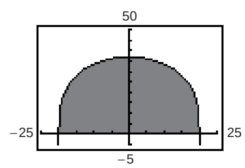
$$M_y = \rho \int_0^5 10x^2\sqrt{125 - x^3}\,dx = -\frac{10\rho}{3}\int_0^5 \sqrt{125 - x^3}(-3x^2)\,dx = \frac{12{,}500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is $(3.0, 126.0)$.

31. $m = \rho \int_{-20}^{20} 5\sqrt[3]{400 - x^2}\,dx \approx 1239.76\rho$



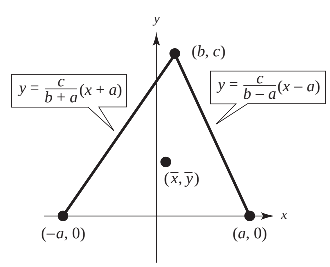
$$M_x = \rho \int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2}\left(5\sqrt[3]{400 - x^2}\right) dx$$

$$= \frac{25\rho}{2}\int_{-20}^{20} (400 - x^2)^{2/3}\,dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 16.2)$.

33. $A = \frac{1}{2}(2a)c = ac$



$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b - a}{c}y + a\right)^2 - \left(\frac{b + a}{c}y - a\right)^2\right] dy$$

$$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right] dy$$

$$= \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b - a}{c}y + a\right) - \left(\frac{b + a}{c}y - a\right)\right] dy$$

$$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right) dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right) dy$$

$$= \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

From elementary geometry, $(b/3, c/3)$ is the point of intersection of the medians.

35. $A = \frac{c}{2}(a+b)$

$\frac{1}{A} = \frac{2}{c(a+b)}$

$\bar{x} = \frac{2}{c(a+b)}\int_0^c x\left(\frac{b-a}{c}x + a\right) dx = \frac{2}{c(a+b)}\int_0^c \left(\frac{b-a}{c}x^2 + ax\right) dx = \frac{2}{c(a+b)}\left[\frac{b-a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$

$= \frac{2}{c(a+b)}\left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a+b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}$

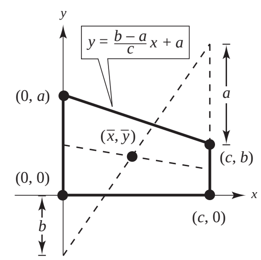
$\bar{y} = \frac{2}{c(a+b)}\frac{1}{2}\int_0^c \left(\frac{b-a}{c}x + a\right)^2 dx = \frac{1}{c(a+b)}\int_0^c \left[\left(\frac{b-a}{c}\right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2\right] dx$

$= \frac{1}{c(a+b)}\left[\left(\frac{b-a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b-a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a+b)}\left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c\right]$

$= \frac{1}{3c(a+b)}[(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$

$= \frac{1}{3(a+b)}[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a+b)}$

Thus, $(\bar{x}, \bar{y}) = \left(\frac{(a+2b)c}{3(a+b)}, \frac{a^2+ab+b^2}{3(a+b)}\right)$.



The one line passes through $(0, a/2)$ and $(c, b/2)$. It's equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$. The other line passes through $(0, -b)$ and $(c, a+b)$. It's equation is $y = \frac{a+2b}{c}x - b$. $(\bar{x}, \bar{y})$ is the point of intersection of these two lines.

37. $\bar{x} = 0$ by symmetry.

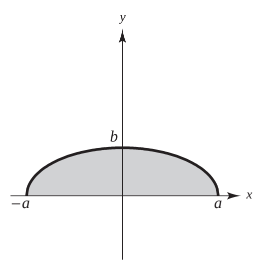
$A = \frac{1}{2}\pi ab$

$\frac{1}{A} = \frac{2}{\pi ab}$

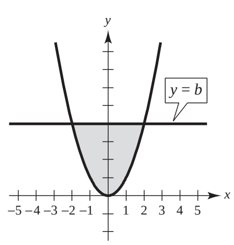
$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$

$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left[\frac{4a^3}{3}\right] = \frac{4b}{3\pi}$

$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$



39. (a)



(b) $\bar{x} = 0$ by symmetry.

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\, dx = 0$ because $bx - x^3$ is odd.

(d) $\bar{y} > \frac{b}{2}$ since there is more area above $y = \frac{b}{2}$ than below.

(e) $M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b+x^2)(b-x^2)}{2}\, dx$

$= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2}\, dx = \frac{1}{2}\left[b^2x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}}$

$= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$

$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2)\, dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}}$

$= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$

$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b$

41. (a) $\bar{x} = 0$ by symmetry.

$$A = 2\int_0^{40} f(x)\,dx = \frac{2(40)}{3(4)}[30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2}\,dx = \frac{40}{3(4)}[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3}(7216) = \frac{72{,}160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72{,}160/3}{5560/3} = \frac{72{,}160}{5560} \approx 12.98$$

$(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$ (Use nine data points.)

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{23{,}697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

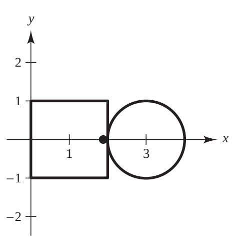
43. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$$



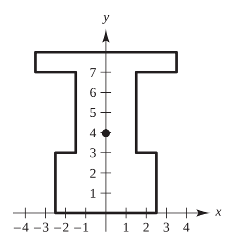
45. Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, $(0, 5)$, and $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right) \approx (0, 3.97)$$



47. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$\bar{y} = 0$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

49. $r = 5$ is distance between center of circle and y-axis.
$A \approx \pi(4)^2 = 16\pi$ is area of circle. Hence,

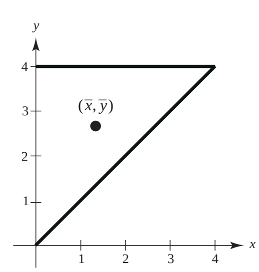
$V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14.$

51. $A = \dfrac{1}{2}(4)(4) = 8$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



53. $m = m_1 + \cdots + m_n$

$M_y = m_1x_1 + \cdots + m_nx_n$

$M_x = m_1y_1 + \cdots + m_ny_n$

$\bar{x} = \dfrac{M_y}{m}, \bar{y} = \dfrac{M_x}{m}$

55. (a) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$

(b) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$

(c) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$

(d) No

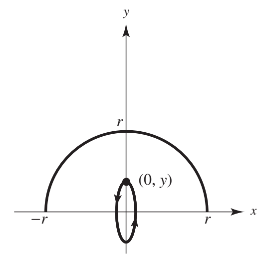
57. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius y.

$$d = 2\pi y$$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.



59. $A = \displaystyle\int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$

$m = \rho A = \dfrac{\rho}{n+1}$

$M_x = \dfrac{\rho}{2}\displaystyle\int_0^1 (x^n)^2\,dx = \left[\frac{\rho}{2}\cdot\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$

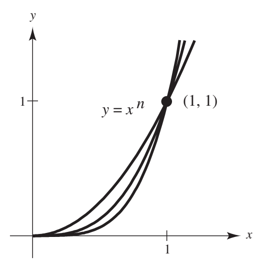
$M_y = \rho\displaystyle\int_0^1 x(x^n)\,dx = \left[\rho\cdot\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{n+1}{n+2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{n+1}{2(2n+1)} = \dfrac{n+1}{4n+2}$

Centroid: $\left(\dfrac{n+1}{n+2}, \dfrac{n+1}{4n+2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \frac{1}{4}\right)$. The graph approaches the x-axis and the line $x = 1$ as $n \to \infty$.



Section 7.7 Fluid Pressure and Fluid Force

1. $F = PA = [62.4(5)](3) = 936$ lb

3. $F = 62.4(h + 2)(6) - (62.4)(h)(6)$

$= 62.4(2)(6) = 748.8$ lb

5. $h(y) = 3 - y$

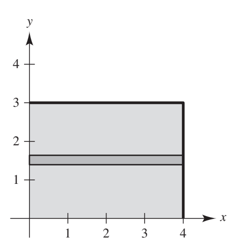
$L(y) = 4$

$F = 62.4\displaystyle\int_0^3 (3 - y)(4)\,dy$

$= 249.6\displaystyle\int_0^3 (3 - y)\,dy$

$= 249.6\left[3y - \dfrac{y^2}{2}\right]_0^3$

$= 1123.2$ lb



7. $h(y) = 3 - y$

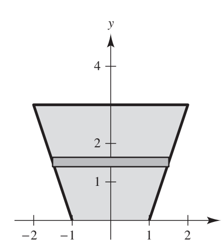
$L(y) = 2\left(\dfrac{y}{3} + 1\right)$

$F = 2(62.4)\displaystyle\int_0^3 (3 - y)\left(\frac{y}{3} + 1\right)dy$

$= 124.8\displaystyle\int_0^3 \left(3 - \frac{y^2}{3}\right)dy$

$= 124.8\left[3y - \dfrac{y^3}{9}\right]_0^3$

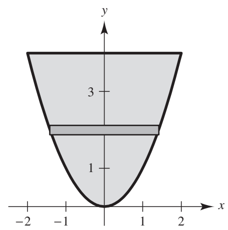
$= 748.8$ lb



9. $h(y) = 4 - y$

$L(y) = 2\sqrt{y}$

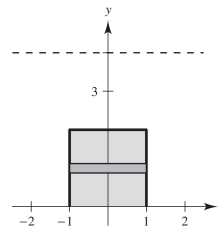
$$F = 2(62.4)\int_0^4 (4 - y)\sqrt{y}\,dy$$
$$= 124.8\int_0^4 (4y^{1/2} - y^{3/2})\,dy$$
$$= 124.8\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4$$
$$= 1064.96 \text{ lb}$$



11. $h(y) = 4 - y$

$L(y) = 2$

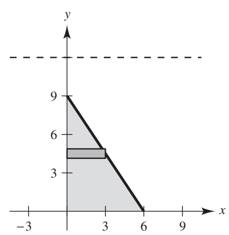
$$F = 9800\int_0^2 2(4 - y)\,dy$$
$$= 9800\Big[8y - y^2\Big]_0^2 = 117{,}600 \text{ newtons}$$



13. $h(y) = 12 - y$

$L(y) = 6 - \dfrac{2y}{3}$

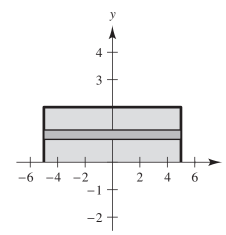
$$F = 9800\int_0^9 (12 - y)\left(6 - \frac{2y}{3}\right)dy$$
$$= 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_0^9$$
$$= 2{,}381{,}400 \text{ newtons}$$



15. $h(y) = 2 - y$

$L(y) = 10$

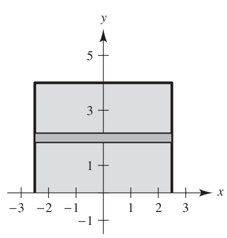
$$F = 140.7\int_0^2 (2 - y)(10)\,dy$$
$$= 1407\int_0^2 (2 - y)\,dy$$
$$= 1407\left[2y - \frac{y^2}{2}\right]_0^2 = 2814 \text{ lb}$$



17. $h(y) = 4 - y$

$L(y) = 6$

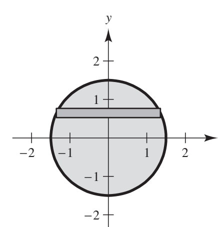
$$F = 140.7\int_0^4 (4 - y)(6)\,dy$$
$$= 844.2\int_0^4 (4 - y)\,dy$$
$$= 844.2\left[4y - \frac{y^2}{2}\right]_0^4 = 6753.6 \text{ lb}$$



19. $h(y) = -y$

$L(y) = 2\left(\dfrac{1}{2}\right)\sqrt{9 - 4y^2}$

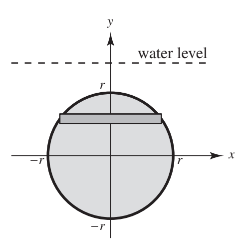
$$F = 42\int_{-3/2}^0 (-y)\sqrt{9 - 4y^2}\,dy$$
$$= \frac{42}{8}\int_{-3/2}^0 (9 - 4y^2)^{1/2}(-8y)\,dy$$
$$= \left[\left(\frac{21}{4}\right)\left(\frac{2}{3}\right)(9 - 4y^2)^{3/2}\right]_{-3/2}^0 = 94.5 \text{ lb}$$



21. $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$$F = w\int_{-r}^r (k - y)\sqrt{r^2 - y^2}\,(2)\,dy$$
$$= w\left[2k\int_{-r}^r \sqrt{r^2 - y^2}\,dy + \int_{-r}^r \sqrt{r^2 - y^2}\,(-2y)\,dy\right]$$



The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r.

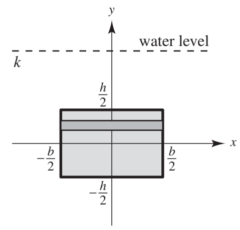
$$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$$

23. $h(y) = k - y$

$L(y) = b$

$$F = w\int_{-h/2}^{h/2} (k - y)b\,dy$$

$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



25. From Exercise 23:

$F = 64(15)(1)(1) = 960$ lb

27. $h(y) = 4 - y$

$$F = 62.4\int_0^4 (4 - y)L(y)\,dy$$

Using Simpson's Rule with $n = 8$ we have:

$$F \approx 62.4\left(\frac{4 - 0}{3(8)}\right)[0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0]$$

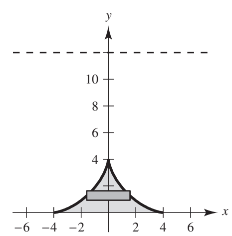
$= 3010.8$ lb

29. $h(y) = 12 - y$

$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$

$$F = 62.4\int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2}\,dy$$

≈ 6448.73 lb



31. (a) If the fluid force is one-half of 1123.2 lb, and the height of the water is b, then

$h(y) = b - y$

$L(y) = 4$

$$F = 62.4\int_0^b (b - y)(4)\,dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y)\,dy = 2.25$$

$$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

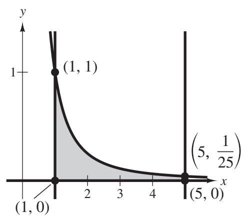
$$b^2 = 4.5 \implies b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

33. $F = Fw = w\displaystyle\int_c^d h(y)L(y)\,dy$, see page 508.

Review Exercises for Chapter 7

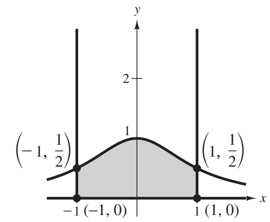
1. $A = \int_1^5 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$



3. $A = \int_{-1}^{1} \frac{1}{x^2+1}\,dx$

$= \left[\arctan x\right]_{-1}^{1}$

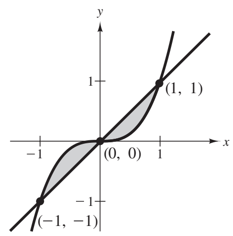
$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$



5. $A = 2\int_0^1 (x - x^3)\,dx$

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

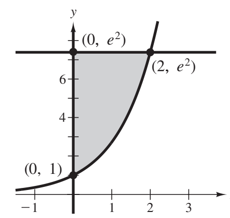
$= \frac{1}{2}$



7. $A = \int_0^2 (e^2 - e^x)\,dx$

$= \left[xe^2 - e^x\right]_0^2$

$= e^2 + 1$

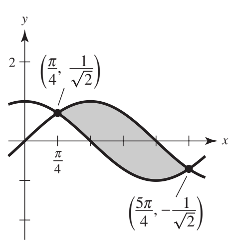


9. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

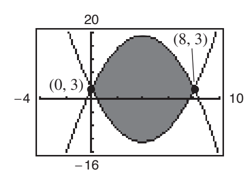
$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$



11. $A = \int_0^8 \left[(3 + 8x - x^2) - (x^2 - 8x + 3)\right]dx$

$= \int_0^8 (16x - 2x^2)\,dx$

$= \left[8x^2 - \frac{2}{3}x^3\right]_0^8 = \frac{512}{3} \approx 170.667$

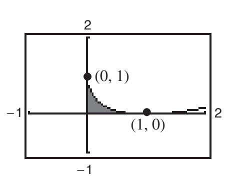


13. $y = \left(1 - \sqrt{x}\right)^2$

$A = \int_0^1 \left(1 - \sqrt{x}\right)^2 dx$

$= \int_0^1 (1 - 2x^{1/2} + x)\,dx$

$= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6} \approx 0.1667$



15. $x = y^2 - 2y \implies x + 1 = (y-1)^2 \implies y = 1 \pm \sqrt{x+1}$

$A = \int_{-1}^{0} \left[\left(1 + \sqrt{x+1}\right) - \left(1 - \sqrt{x+1}\right)\right]dx$

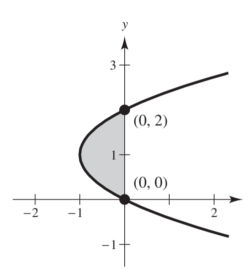
$= \int_{-1}^{0} 2\sqrt{x+1}\,dx$

$A = \int_0^2 \left[0 - (y^2 - 2y)\right]dy$

$= \int_0^2 (2y - y^2)\,dy$

$= \left[y^2 - \frac{1}{3}y^3\right]_0^2$

$= \frac{4}{3}$



17. $A = \int_0^2 \left[1 - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [1 - (x - 2)]\, dx$

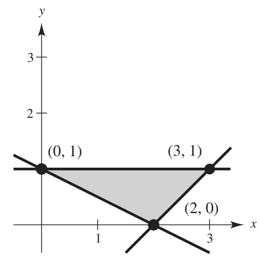
$= \int_0^2 \frac{x}{2}\, dx + \int_2^3 (3 - x)\, dx$

$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$

$y = x - 2 \Rightarrow x = y + 2, y = 1$

$A = \int_0^1 [(y + 2) - (2 - 2y)]\, dy$

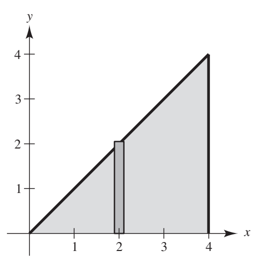
$= \int_0^1 3y\, dy = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$



19. Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

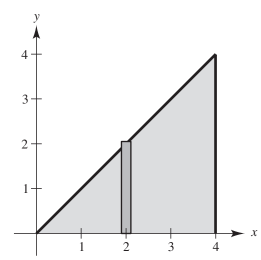
21. (a) **Disk**

$V = \pi \int_0^4 x^2\, dx = \left[\frac{\pi x^3}{3}\right]_0^4 = \frac{64\pi}{3}$



(b) **Shell**

$V = 2\pi \int_0^4 x^2\, dx = \left[\frac{2\pi}{3}x^3\right]_0^4 = \frac{128\pi}{3}$

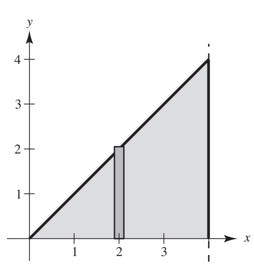


(c) **Shell**

$V = 2\pi \int_0^4 (4 - x)x\, dx$

$= 2\pi \int_0^4 (4x - x^2)\, dx$

$= 2\pi \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \frac{64\pi}{3}$

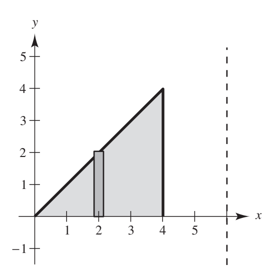


(d) **Shell**

$V = 2\pi \int_0^4 (6 - x)x\, dx$

$= 2\pi \int_0^4 (6x - x^2)\, dx$

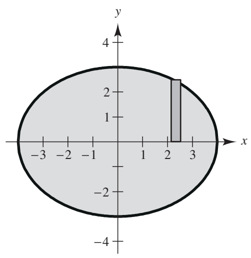
$= 2\pi \left[3x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{160\pi}{3}$



23. (a) **Shell**

$$V = 4\pi \int_0^4 x\left(\frac{3}{4}\right)\sqrt{16 - x^2}\, dx$$

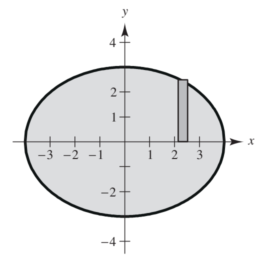
$$= \left[3\pi\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16 - x^2)^{3/2}\right]_0^4 = 64\pi$$



(b) **Disk**

$$V = 2\pi \int_0^4 \left[\frac{3}{4}\sqrt{16 - x^2}\right]^2 dx$$

$$= \frac{9\pi}{8}\left[16x - \frac{x^3}{3}\right]_0^4 = 48\pi$$



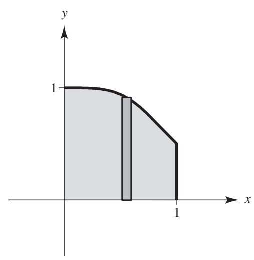
25. Shell

$$V = 2\pi \int_0^1 \frac{x}{x^4 + 1}\, dx$$

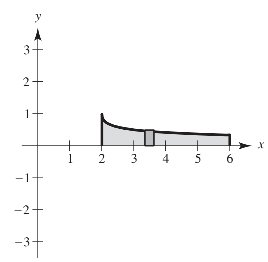
$$= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1}\, dx$$

$$= \left[\pi \arctan(x^2)\right]_0^1$$

$$= \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$



27. Shell: $V = 2\pi \int_2^6 \frac{x}{1 + \sqrt{x - 2}}\, dx$



$u = \sqrt{x - 2}$

$x = u^2 + 2$

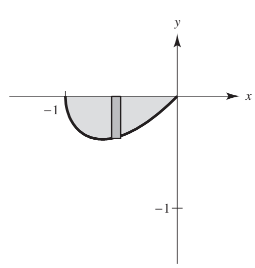
$dx = 2u\, du$

$$V = 2\pi \int_2^6 \frac{x}{1 + \sqrt{x - 2}}\, dx = 4\pi \int_0^2 \frac{(u^2 + 2)u}{1 + u}\, du$$

$$= 4\pi \int_0^2 \frac{u^3 + 2u}{1 + u}\, du = 4\pi \int_0^2 \left(u^2 - u + 3 - \frac{3}{1 + u}\right) du$$

$$= 4\pi\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3\ln(1 + u)\right]_0^2 = \frac{4\pi}{3}(20 - 9\ln 3) \approx 42.359$$

29. Since $y \le 0$, $A = -\int_{-1}^0 x\sqrt{x + 1}\, dx$.



$u = x + 1$

$x = u - 1$

$dx = du$

$$A = -\int_0^1 (u - 1)\sqrt{u}\, du = -\int_0^1 (u^{3/2} - u^{1/2})\, du$$

$$= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^1 = \frac{4}{15}$$

31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disk:
$$\pi\int_{-3}^{y_0} \frac{16}{9}(9 - y^2)\,dy = 16\pi$$
$$\frac{1}{9}\int_{-3}^{y_0} (9 - y^2)\,dy = 1$$
$$\left[9y - \frac{1}{3}y^3\right]_{-3}^{y_0} = 9$$
$$\left(9y_0 - \frac{1}{3}y_0^3\right) - (-27 + 9) = 9$$
$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ ft.

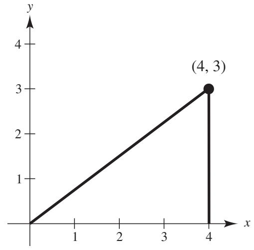
33.
$$f(x) = \frac{4}{5}x^{5/4}$$
$$f'(x) = x^{1/4}$$
$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$
$$u = 1 + \sqrt{x}$$
$$x = (u - 1)^2$$
$$dx = 2(u - 1)\,du$$
$$s = \int_0^4 \sqrt{1 + \sqrt{x}}\,dx = 2\int_1^3 \sqrt{u}(u - 1)\,du$$
$$= 2\int_1^3 (u^{3/2} - u^{1/2})\,du$$
$$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^3 = \frac{4}{15}\left[u^{3/2}(3u - 5)\right]_1^3$$
$$= \frac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$$

35. $y = 300\cosh\left(\frac{x}{2000}\right) - 280,\ -2000 \le x \le 2000$

$$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$$
$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\,dx$$
$$= \frac{1}{20}\int_{-2000}^{2000} \sqrt{400 + 9\sinh^2\left(\frac{x}{2000}\right)}\,dx$$
≈ 4018.2 ft (by Simpson's Rule or graphing utility)

37. $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$
$$1 + (y')^2 = \frac{25}{16}$$
$$S = 2\pi\int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



39. $F = kx$

$$4 = k(1)$$
$$F = 4x$$
$$W = \int_0^5 4x\,dx = \left[2x^2\right]_0^5$$
$$= 50 \text{ in.} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

41. Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $175 - y$

$$W = \frac{62.4\pi}{9}\int_0^{150}(175 - y)\,dy = \frac{62.4\pi}{9}\left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104{,}000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

43. Weight of section of chain: $5\,\Delta x$

Distance moved: $10 - x$

$$W = 5\int_0^{10}(10 - x)\,dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10} = 250 \text{ ft} \cdot \text{lb}$$

45. $W = \int_a^b F(x)\,dx$

$$80 = \int_0^4 ax^2\,dx = \frac{ax^3}{3}\Big]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

47. $A = \int_0^a \left(\sqrt{a} - \sqrt{x}\right)^2 dx = \int_0^a \left(a - 2\sqrt{a}x^{1/2} + x\right) dx = \left[ax - \frac{4}{3}\sqrt{a}x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$

$$\frac{1}{A} = \frac{6}{a^2}$$

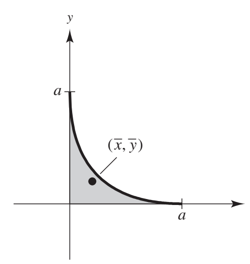
$$\bar{x} = \frac{6}{a^2}\int_0^a x\left(\sqrt{a} - \sqrt{x}\right)^2 dx = \frac{6}{a^2}\int_0^a \left(ax - 2\sqrt{a}x^{3/2} + x^2\right) dx = \frac{a}{5}$$

$$\bar{y} = \left(\frac{6}{a^2}\right)\frac{1}{2}\int_0^a \left(\sqrt{a} - \sqrt{x}\right)^4 dx$$

$$= \frac{3}{a^2}\int_0^a \left(a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2\right) dx$$

$$= \frac{3}{a^2}\left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



49. By symmetry, $x = 0$.

$$A = 2\int_0^1 (a^2 - x^2)\,dx = 2\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4a^3}{3}$$

$$\frac{1}{A} = \frac{3}{4a^3}$$

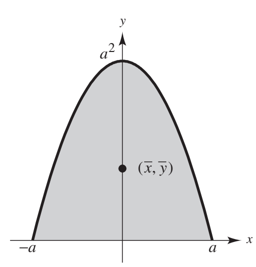
$$\bar{y} = \left(\frac{3}{4a^3}\right)\frac{1}{2}\int_{-a}^a (a^2 - x^2)^2\,dx$$

$$= \frac{6}{8a^3}\int_0^a (a^4 - 2a^2x^2 + x^4)\,dx$$

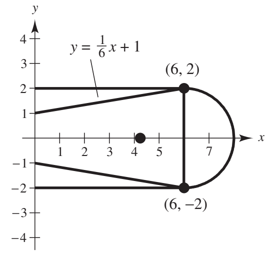
$$= \frac{6}{8a^3}\left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5\right]_0^a$$

$$= \frac{6}{8a^3}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5}\right)$$



51. $\bar{y} = 0$ by symmetry.



For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$M_y = \rho \int_0^6 x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right] dx$$

$$= \rho \int_0^6 \left(\frac{1}{3}x^2 + 2x\right) dx = \rho\left[\frac{x^3}{9} + x^2\right]_0^6 = 60\rho$$

For the semicircle:

$$m = \left(\frac{1}{2}\right)(\pi)(2)^2\rho = 2\pi\rho$$

$$M_y = \rho \int_6^8 x\left[\sqrt{4 - (x-6)^2} - \left(-\sqrt{4 - (x-6)^2}\right)\right] dx = 2\rho \int_6^8 x\sqrt{4 - (x-6)^2}\, dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$M_y = 2\rho \int_0^2 (u + 6)\sqrt{4 - u^2}\, du = 2\rho \int_0^2 u\sqrt{4 - u^2}\, du + 12\rho \int_0^2 \sqrt{4 - u^2}\, du$$

$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(4 - u^2)^{3/2}\right]_0^2 + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3}$$

Thus, we have:

$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4 + 9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3} \cdot \frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$.

53. Let D = surface of liquid; ρ = weight per cubic volume.

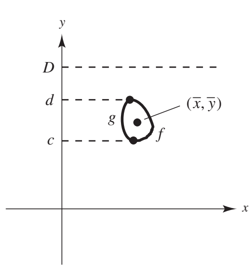
$$F = \rho \int_c^d (D - y)[f(y) - g(y)]\, dy$$

$$= \rho\left[\int_c^d D[f(y) - g(y)]\, dy - \int_c^d y[f(y) - g(y)]\, dy\right]$$

$$= \rho\left[\int_c^d [f(y) - g(y)]\, dy\right]\left[D - \frac{\int_c^d y[f(y) - g(y)]\, dy}{\int_c^d [f(y) - g(y)]\, dy}\right]$$

$$= \rho(\text{Area})(D - \bar{y})$$

$$= \rho(\text{Area})(\text{depth of centroid})$$



Problem Solving for Chapter 7

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2)\,dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c\to 0^+} \frac{T}{R} = \lim_{c\to 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

3. (a) $\frac{1}{2}V = \int_0^1 \left[\pi\left(2 + \sqrt{1-y^2}\right)^2 - \pi\left(2 - \sqrt{1-y^2}\right)^2\right] dy$

$$= \pi\int_0^1 \left[\left(4 + 4\sqrt{1-y^2} + (1-y^2)\right) - \left(4 - 4\sqrt{1-y^2} + (1-y^2)\right)\right] dy$$

$$= 8\pi\int_0^1 \sqrt{1-y^2}\,dy \quad \text{(Integral represents 1/4 (area of circle))}$$

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2 \implies V = 4\pi^2$$

(b) $(x - R)^2 + y^2 = r^2 \implies x = R \pm \sqrt{r^2 - y^2}$

$$\frac{1}{2}V = \int_0^r \left[\pi\left(R + \sqrt{r^2-y^2}\right)^2 - \pi\left(R - \sqrt{r^2-y^2}\right)^2\right] dy$$

$$= \pi\int_0^r 4R\sqrt{r^2-y^2}\,dy$$

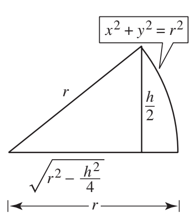
$$= \pi(4R)\frac{1}{4}\pi r^2 = \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

5. $V = 2(2\pi)\int_{\sqrt{r^2-(h^2/4)}}^{r} x\sqrt{r^2-x^2}\,dx$

$$= -2\pi\left[\frac{2}{3}(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2-(h^2/4)}}^{r}$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!$$



7. (a) Tangent at A: $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$
$$y = 3x - 2$$

To find point B: $x^3 = 3x - 2$

$$x^3 - 3x + 2 = 0$$
$$(x - 1)^2(x + 2) = 0 \implies B = (-2, -8)$$

Tangent at B: $y = x^3, y' = 3x^2$

$$y + 8 = 12(x + 2)$$
$$y = 12x + 16$$

To find point C: $x^3 = 12x + 16$

$$x^3 - 12x - 16 = 0$$
$$(x + 2)^2(x - 4) = 0 \implies C = (4, 64)$$

Area of $R = \int_{-2}^{1} (x^3 - 3x + 2)\, dx = \frac{27}{4}$

Area of $S = \int_{-2}^{4} (12x + 16 - x^3)\, dx = 108$

Area of $S = 16$(area of R) $\left[\frac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x - a)^2(x + 2a) = 0$$
$$\implies B = (-2a, -8a^3)$$

Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x + 2a)^2(x - 4a) = 0$$
$$\implies C = (4a, 64a^3)$$

Area of $R = \int_{-2a}^{a} [x^3 - 3a^2x + 2a^3]\, dx = \frac{27}{4}a^4$

Area of $S = \int_{-2a}^{4a} [12a^2x + 16a^3 - x^3]\, dx = 108a^4$

Area of $S = 16$(area of R)

9. $s(x) = \int_{\alpha}^{x} \sqrt{1 + f'(t)^2}\, dt$

(a) $s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\, dx$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_{1}^{x} \sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2}\, dt = \int_{1}^{x} \sqrt{1 + \frac{9}{4}t}\, dt$

(d) $s(2) = \int_{1}^{2} \sqrt{1 + \frac{9}{4}t}\, dt = \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_1^2 = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$

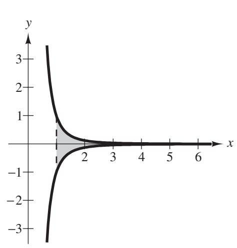
This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

11. (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_{1}^{6} x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right) dx = \int_{1}^{6} \frac{2}{x^2}\, dx = \left[-2\frac{1}{x}\right]_1^6 = \frac{5}{3}$$

$$m = 2\int_{1}^{6} \frac{1}{x^3}\, dx = \left[-\frac{1}{x^2}\right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$



(b) $m = 2\int_{1}^{b} \frac{1}{x^3}\, dx = \frac{b^2 - 1}{b^2}$

$$M_y = 2\int_{1}^{6} \frac{1}{x^2}\, dx = \frac{2(b - 1)}{b}$$

$$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0\right)$$

(c) $\lim_{b\to\infty} \bar{x} = \lim_{b\to\infty} \frac{2b}{b + 1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$

13. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

15. Point of equilibrium: $50 - 0.5x = 0.125x$

$x = 80, p = 10$

$(P_0, x_0) = (10, 80)$

Consumer surplus $= \int_0^{80} [(50 - 0.5x) - 10]\, dx = 1600$

Producer surplus $= \int_0^{80} [10 - 0.125x]\, dx = 400$

17. We use Exercise 23, Section 7.7, which gives $F = wkhb$ for a rectangle plate.

Wall at shallow end

From Exercise 23: $F = 62.4(2)(4)(20) = 9984$ lb

Wall at deep end

From Exercise 23: $F = 62.4(4)(8)(20) = 39{,}936$ lb

Side wall

From Exercise 23: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$$F_2 = 62.4 \int_0^4 (8 - y)(10y)\, dy$$

$$= 624 \int_0^4 (8y - y^2)\, dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$$

$$= 26{,}624 \text{ lb}$$

Total force: $F_1 + F_2 = 46{,}592$ lb