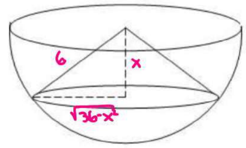
<u>Open-Ended Practice Problems</u>:

Please show all work for each problem. Work should be neat, organized and flow in a logical format. Leave all answers as exact values unless otherwise specified.

1. A right circular cone is inscribed in a semi-sphere with a radius of 6 inches. What would be the dimensions of the cone with maximum volume and what is the maximum volume?



$V = \frac{1}{3}\pi\left(\sqrt{36-x^2}\right)^2 x = \frac{\pi}{3}(36x - x^3)$

$V'(x) = \frac{\pi}{3}(36 - 3x^2)$

$x = \pm 2\sqrt{3}$

$V'(x)$: $+$ | $-$

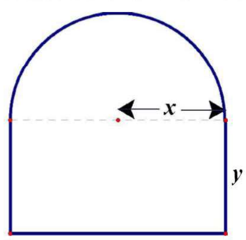
$V(x)$: 0 ↗ $2\sqrt{3}$ ↘ 6

$r = \sqrt{36-12} = 2\sqrt{6}$

$V = \frac{\pi}{3}(2\sqrt{6})^2(2\sqrt{3}) = 16\pi\sqrt{3}$

Height: $2\sqrt{3}$ in
radius: $2\sqrt{6}$ in
Volume: $16\pi\sqrt{3}$ in^3

2. A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of the Norman window of maximum area if the total perimeter of the outside of the window is 60 feet.



$P = 2y + 2x + \pi x = 60$

$y = 30 - x - \frac{\pi}{2}x$

$A = \frac{1}{2}\pi x^2 + 2x\left(30 - x - \frac{\pi}{2}x\right)$

$A = \frac{1}{2}\pi x^2 + 60x - 2x^2 - \pi x^2$

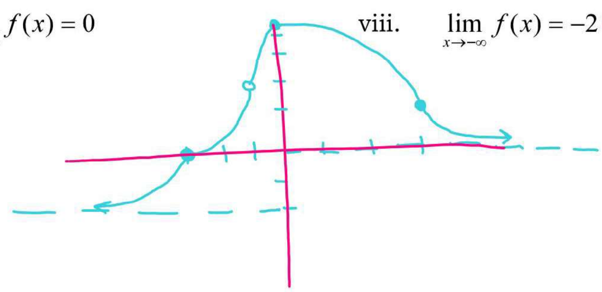
$A' = \pi x + 60 - 4x - 2\pi x = -\pi x - 4x + 60 = 0$

$x = \frac{60}{\pi + 4}$

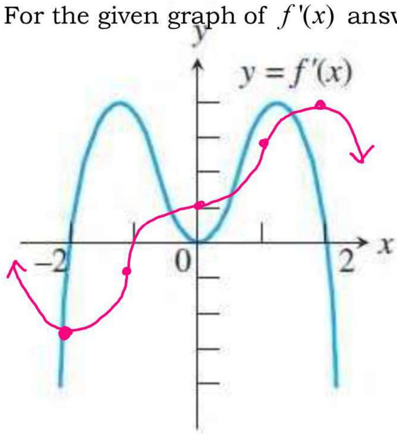
Width: $\frac{120}{\pi+4}$ ft
Length: $\frac{60}{\pi+4}$ ft
Arc Length: $\frac{60\pi}{\pi+4}$ ft

3. Sketch a graph that has all of the following characteristics:

i.	$f(-1)$ DNE ✓	ii.	$f'(0) = 0$ & $f''(0) < 0$ ✓
iii.	$f' > 0$ on $(-\infty,-1)\cup(-1,0)$	iv.	$f'' > 0$ on $(-\infty,-1)\cup(3,\infty)$ ✓
v.	$f(3) = 1$, $f(0) = 5$, $f(-3) = 0$ ✓	vi.	$f'' < 0$ on $(-1,3)$ ✓
vii.	$\lim\limits_{x\to\infty} f(x) = 0$	viii.	$\lim\limits_{x\to-\infty} f(x) = -2$ ✓
ix.	$\lim\limits_{x\to-1} f(x) = 2$ ✓		



4. For the given graph of $f'(x)$ answer the following questions. Then, sketch a possible graph of $f(x)$.



a) intervals on which $f(x)$ is increasing $(-2,0) \cup (0,2)$ $f'(x)>0$

b) intervals on which $f(x)$ is decreasing $(-\infty,-2) \cup (2,\infty)$ $f'(x)<0$

c) relative maximum of function at $x =$ 2 $f'(x)>0$ to $f'(x)<0$

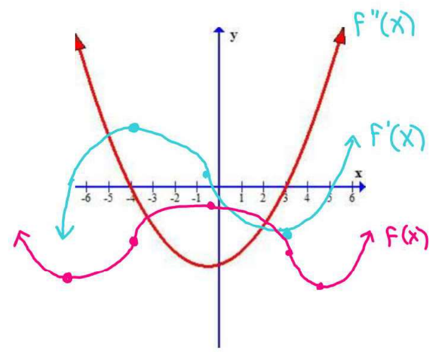
d) relative minimum of function at $x =$ -2 $f'(x)<0$ to $f'(x)>0$

e) points of inflection for $f(x)$ at $x =$ 1, 0, -1 $f'(x)$ extrema

f) intervals on which $f(x)$ is concave up $(-\infty,-1) \cup (0,1)$ $f'(x)$ Inc

g) intervals on which $f(x)$ is concave down $(-1,0) \cup (1,\infty)$ $f'(x)$ Dec

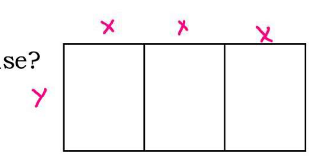
5. Given the graph of $f''(x)$, sketch the graph of $f(x)$ and $f'(x)$.



6. Consider a rectangular warehouse consisting of three separate spaces of equal size, as pictured below. Assume that the wall materials cost \$200 per linear foot and the company allocates \$2,400,000 for the project. *Hint: \$200 per linear foot means for every foot of perimeter it costs \$200 for wall material.*

a) Which dimensions maximize the total area of the warehouse?

b) What is the area of each compartment in this case?



$A = 3xy$ $\quad P = 4y + 6x \Rightarrow 2{,}400{,}000 = 200(4y+6x)$

$y = 3000 - \frac{3}{2}x$

$A = 3x(3000 - \frac{3}{2}x)$

$A = 9000x - \frac{9}{2}x^2$

$A' = 9000 - 9x = 0$

$x = 1{,}000$

$+$ $-$ $f'(x)$

0 ↗ 10 ↘ $f(x)$

$x = 1{,}000$ yields a max area since $f'(x)>0$ to $f'(x)<0$

Warehouse:

width: 3,000 ft

height: 1,500 ft

Compartment:

area: 1,500,000 ft^2

7. Determine whether Rolle's Theorem can be applied to the function $f(x)=\cos(2x)$ on the interval $\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$. If Rolle's Theorem can be applied, find all values of x that satisfy Rolle's Theorem. Explain what is graphically happening at this value of *c*.

$f(x)$ is continuous on $\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$ and differentiable on $\left(\frac{\pi}{3},\frac{2\pi}{3}\right)$

$f\left(\frac{\pi}{3}\right) = -\frac{1}{2} = f\left(\frac{2\pi}{3}\right)$

$f'(x) = -2\sin(2x) = 0$

$2x = \pi$

$x = \frac{\pi}{2}$

At $x=\frac{\pi}{2}$, there is a horizontal tangent line and an extrema value. At this point the tangent line is parallel to the secant line on the interval $\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$.

8. Let f be the function given by $f(x)=3\ln(x^2+2)-2x$ with domain $[-2,4]$.

a. Find the coordinate of each relative maximum point and each relative minimum point of *f*. Justify your answer.

b. Find the *x*-coordinate of each point of inflection of the graph of *f*.

c. Find the absolute maximum value of *f(x)*.

A. $f'(x) = \frac{6x}{x^2+2} - 2 = \frac{-2(x-1)(x-2)}{x^2+2}$ C.P. $x = 1, 2$

$f'(x)$: $-$ $+$ $-$

$f(x)$: -2 ↘ 1 ↗ 2 ↘ 4

$(1, 3\ln(3)-2)$ rel. min since $f'(x)<0$ to $f'(x)>0$

$(2, 3\ln(6)-4)$ rel. max since $f'(x)>0$ to $f'(x)<0$

B. $f''(x) = \frac{-6(x^2-2)}{(x^2+2)^2}$

P.P.O.I.

$f''(x)$: $-$ $+$ $-$

$f(x)$: -2 $-\sqrt{2}$ $\sqrt{2}$ 4

P.O.I. $x = \pm\sqrt{2}$

C. $f(-2) = 3\ln 6 + 4$ $(-2, 3\ln 6+4)$ Abs. max

$f(2) = 3\ln 6 - 4$ $(2, 3\ln 6 - 4)$ Abs. min

9. Two points are located on $f(x)=\cos(x)$ on the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. These two points are connected and, along with the *x*-axis, form opposite sides of a rectangle. What length and width should the rectangle have so that its area is a maximum? What is the maximum area? Round answers to 3 decimal places.

$A = 2xy = 2x\cos x$

$A'(x) = 2\cos x - 2x\sin x = 0$

$\cancel{x \approx -.860}$ $x \approx .860$

$A''(x) = 2\sin x - 2\sin x - 2x\cos x = -2x\cos x$

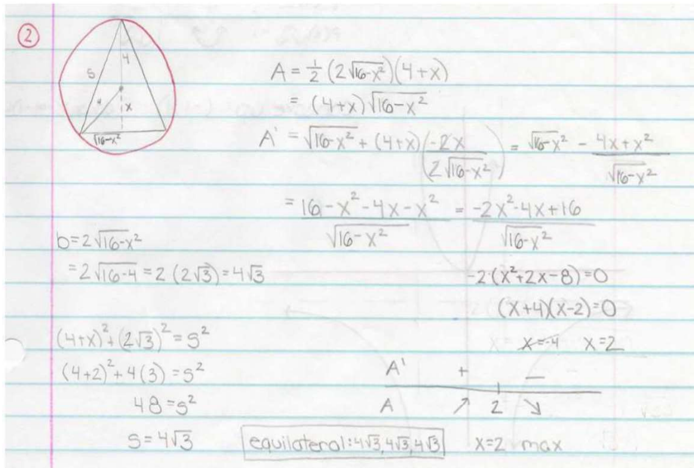
$\left.\begin{matrix} A''(.860) < 0 \\ A'(.860) = 0 \end{matrix}\right\}$ $x = .860$ yields a max

width: 1.720

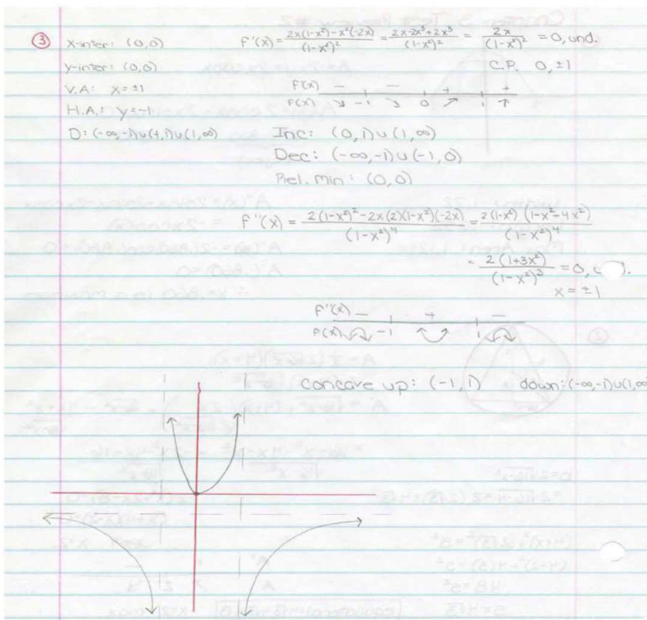
length: .652

max Area: 1.121

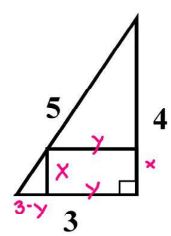
10. Find the side lengths of the isosceles triangle with maximum area that is inscribed in a circle of radius four.



11. Completely analyze and sketch the function $f(x)=\dfrac{x^2}{1-x^2}$.



12. Find the rectangle of maximum area that can be inscribed in a right triangle with legs of length 3in and 4in if the sides of the rectangle are parallel to the legs of the triangle, as in the diagram to the right. Identify the maximum area and the length and width that produce the maximum area.



$A = xy = (4-\frac{4}{3}y)y = 4y - \frac{4}{3}y^2$

$A' = 4 - \frac{8}{3}y = 0$

$y = \frac{3}{2}$ $\quad x = 4 - \frac{4}{3}\left(\frac{3}{2}\right) = 2$

$A''(y) = -\frac{8}{3}$

$\left.\begin{array}{l} A''\left(\frac{3}{2}\right) < 0 \\ A'\left(\frac{3}{2}\right) = 0 \end{array}\right\}$ $y=\frac{3}{2}$ yields a max value

$\frac{x}{3-y} = \frac{4}{3}$

$3x = 12 - 4y$

$x = 4 - \frac{4}{3}y$

$\boxed{2\text{in by } \frac{3}{2}\text{in}}$

13. You are driving in a car traveling at 50 mph and you pass a police car. You don't worry because you are going the speed limit of 50 mph. Four minutes later, you pass a second police car and you are traveling at 50mph. The distance between the two police cars is five miles. The second police car nails you for speeding. Explain how he can use the Mean Value Theorem to prove that you were speeding in complete sentences.

$(0,0)$ min miles $(4,5)$ min miles $\frac{f(4)-f(0)}{4-0} = \frac{5-0}{4-0} = \frac{5}{4}\frac{mi}{min}$ $\frac{5}{4}\frac{mi}{min}\left(\frac{60min}{1\ hr}\right) = 75\ mi/hr$

By the MVT, there is a least one value of c in $(0,4)$ such that $f'(c) = 75$ mi/hr.

14. Somewhere between 0 and $\frac{\pi}{3}$, the derivative of the function $f(x)=\cos(x)$ is guaranteed to equal what value? Justify your answer.

$f(x)$ is continuous on $[0,\frac{\pi}{3}]$

$f(x)$ is differentiable on $(0,\frac{\pi}{3})$

By the MVT, there exists at least one point c such that $f'(x) = \frac{f(\frac{\pi}{3})-f(0)}{\frac{\pi}{3}-0}$.

$\therefore\ f'(x) = \frac{\frac{1}{2}-1}{\frac{\pi}{3}}$

$\therefore\ f'(x) = \frac{-3}{2\pi}$

Multiple Choice Practice Problems:

15. The tangent to the curve of $y = xe^{-x}$ is horizontal when x is equal to

a) 0 $\boxed{\textbf{b)}}$ 1 **c)** -1 **d)** $\frac{1}{e}$ **e)** *none of these*

$y' = e^{-x} - xe^{-x} = 0$

$e^{-x}(1-x) = 0$

$x = 1$

16. The function $f(x) = x^4 - 4x^2$ has

a) one relative minimum and two relative maxima
b) one relative minimum and one relative maximum
c) two relative maxima and one relative minimum
d) two relative minima and no relative maximum
$\boxed{\textbf{e)}}$ two relative minima and one relative maximum

$f'(x) = 4x^3 - 8x = 0$

$4x(x^2-2) = 0$

$x=0\quad x=\pm\sqrt{2}$

$f'(x)$: $-$ | $+$ | $-$ | $+$

$f(x)$: ↘ $-\sqrt{2}$ ↗ 0 ↘ $\sqrt{2}$ ↗

17. The number of inflection points of the curve $f(x) = x^4 - 4x^2$ is

a) 0 **b)** 1 $\boxed{\textbf{c)}}$ 2 **d)** 3 **e)** 4

$f''(x) = 12x^2 - 8 = 0$

$x = \pm\frac{\sqrt{6}}{3}$

$f''(x)$: $+$ | $-$ | $+$

$f(x)$: ∪ $-\frac{\sqrt{6}}{3}$ ∩ $\frac{\sqrt{6}}{2}$ ∪

18. The maximum value of the function $y = -4\sqrt{2-x}$ is

D: $(-\infty, 2]$

$y' = 4\left(\frac{1}{2\sqrt{2-x}}\right) = \frac{2}{\sqrt{2-x}}$

$\boxed{\textbf{a)}}$ 0 **b)** -4 **c)** 2 **d)** -2 **e)** *none of these*

$y(2) = 0$

y' +]
y ↗]

19. The total number of local maximum and minimum points of the function whose derivative, for all x, is given by $f'(x) = x(x-3)^2(x+1)^4$ is

$f'(x)$ − | − | + | +
$f(x)$ ↘ −1 ↘ 0 ↗ 3 ↗

a) 0 $\boxed{\textbf{b)}}$ 1 **c)** 2 **d)** 3 **e)** *none of these*

$y^2 = x^2 - 4$ (6,0) (x,y)

20. The points on the curve $x^2 - y^2 = 4$ closest to the point (6,0) is (are)

$d^2 = (6-x)^2 + (0-y)^2$

$d^2 = (6-x)^2 + x^2 - 4$

a) $(2,0)$ **b)** $(\sqrt{5}, \pm 1)$ $\boxed{\textbf{c)}}$ $(3, \pm\sqrt{5})$ **d)** $(\sqrt{13}, \pm\sqrt{3})$ **e)** *none of these*

$2d\,d' = -2(6-x) + 2x$

$d' = \frac{4x-12}{\sqrt{(6-x)^2 + x^2 - 4}} = 0$

$x = 3$

d' − | +
d ↘ 3 ↗

21. The sum of the squares of two positive numbers is 200. Their maximum product is

$x^2 + y^2 = 200$

$y = \sqrt{200 - x^2}$

$\boxed{\textbf{a)}}$ 100 **b)** $25\sqrt{7}$ **c)** 28 **d)** $24\sqrt{14}$ **e)** *none of these*

$P(x,y) = x\sqrt{200-x^2}$

$P'(x,y) = \sqrt{200-x^2} - \frac{x^2}{\sqrt{200-x^2}} = \frac{200-2x^2}{\sqrt{200-x^2}} = 0$

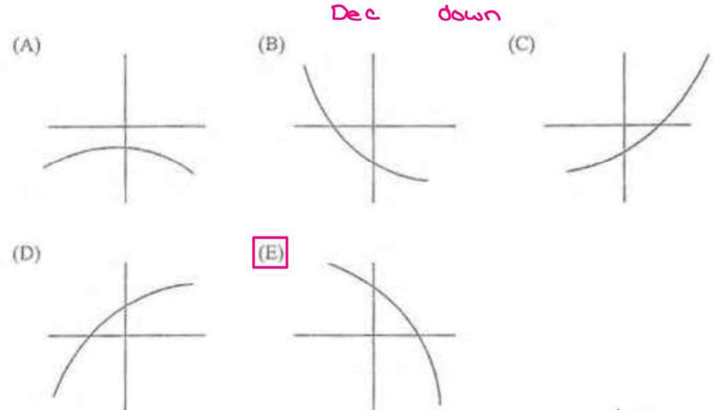
$200 - 2x^2 = 0$

$x = 10$

$P'(x)$ + | −
$P(x)$ ↗ 10 ↘

22. For which curve of f below are both f' and f'' negative?

Dec down



up dec

23. For which curve of f in question #22 is f'' positive but f' negative?

a) Graph A $\boxed{\textbf{b)}}$ Graph B **c)** Graph C **d)** Graph D **e)** Graph E

24. If $y = xe^{-x}$, then at $x = 0$

$y' = e^{-x} - xe^{-x} = 0$
$e^{-x}(1-x) = 0$

$f'(0) = 1 > 0$

$\boxed{\textbf{a)}}$ f is increasing **b)** f is decreasing **c)** f has a relative maximum

d) f has a relative minimum **e)** f' does not exist

25. A local minimum value of the function $y = \dfrac{e^x}{x}$ is

$y' = \dfrac{e^x(x) - e^x}{x^2} = 0$

$e^x(x-1) = 0$

$x = 1$

a) -1 **b)** 1 **c)** $\dfrac{1}{e}$ $\boxed{\textbf{d)}}$ e **e)** 0

y' − | +
y ↘ 1 ↗

$y(1) = e$

26. A function f has a derivative for each x such that $|x| < 2$ and has a local minimum at $(2, -5)$. Which statement below must be true?

$f'(x)$ exists on $(-2, 2)$

~~**a)**~~ $f'(2) = 0$ or $f'(2)$ undefined
~~**b)**~~ f' exists at $x = 2$ $f'(2)$ could be undefined
~~**c)**~~ The graph is concave up at $x = 2$
~~**d)**~~ $f'(x) < 0$ if $x < 2$, $f'(x) > 0$ if $x > 2$
$\boxed{\textbf{e)}}$ None of the preceding are necessarily true

Possible
Not necessarily

27. If $f'(x) = \sqrt[3]{x^2 - 9}$, then $f(x)$ has a relative minimum at $x =$

a) -3 **b)** -2.080 **c)** 0 **d)** 2.080 $\boxed{\textbf{e)}}$ 3

$F'(x)$ + | − | +
$F(x)$ ↗ −3 ↘ 3 ↗

28. A function f is continuous and differentiable on the interval [0, 4], where f' is positive but f'' is negative. Which table could represent points on f?

inc
down

~~(A)~~

x	0	1	2	3	4
y	10	12	14	16	18

constant

~~(B)~~

x	0	1	2	3	4
y	10	12	15	19	24

up

$\boxed{\text{(C)}}$

x	0	1	2	3	4
y	10	18	24	28	30

down * increasing less + less *

~~(D)~~

x	0	1	2	3	4
y	30	28	24	18	10

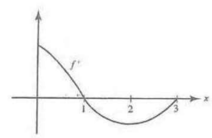
Dec.

~~(E)~~

x	0	1	2	3	4
y	24	19	15	12	10

Dec.

29. Given f' as graphed, which could be the graph of f?

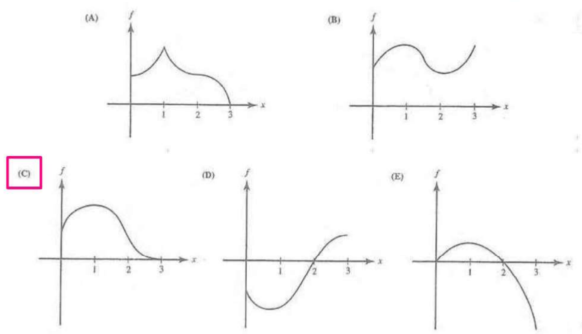


X=1 max

X=2 POI

up (2,3)

down (0,2)



30. The maximum value of $f(x) = x^3 + 3x^2 - 9x - 2$ on the interval [0, 2] is

a. 25 **b.** -7 **c.** -2 $\boxed{\textbf{d.}}$ 0 **e.** 2

$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3)$
$= 3(x+3)(x-1)$

f'(x) [- 1 +]
f(x) 0 ↘ 1 ↗ 2

$f(0) = -2$
$f(2) = 0$

31. Which of the following statements are always true?

~~I.~~ If f has a local minimum at $x = a$, then $f'(a) = 0$. or $f'(a)$ = undefined

II. If $f'(a) = 0$ and $f''(a) = 3$, then f has a local minimum at $x = a$.

~~III.~~ If $f'(a) = 0$ and $f''(a) = 0$, then f does not have a local minimum at $x = a$.

$f(x) = x^4$
$f'(x) = 4x^3$
$f''(x) = 12x^2$

a. I only $\boxed{\textbf{b.}}$ II Only **c.** I & II Only **d.** II & III Only **e.** I, II, & III

$f'(0) = 0$
$f''(0) = 0$
but $x=0$ yields a min

32. If $f'(x) = x\cos(x^2)$ for $-\pi \le x \le \pi$, then the critical point(s) of f on $-\pi < x < \pi$ are $x =$

$x = 0$ $\cos(x^2) = 0$

a. 0 only **b.** 0 & $\frac{\pi}{2}$ **c.** $\frac{\pi}{2}$ & $-\frac{\pi}{2}$ **d.** 0 & $\sqrt{\frac{\pi}{2}}$ $\boxed{\textbf{e.}}$ 0, $\sqrt{\frac{\pi}{2}}$ & $-\sqrt{\frac{\pi}{2}}$

$x^2 = \frac{\pi}{2}$

$x = \pm\sqrt{\frac{\pi}{2}}$

$f'(x) = 4ax^3 + 2bx = 0$

$x(2ax^2 + b) = 0$

$x = 0$

$f''(x) = 12ax^2 + 2b$

$a>0$ + $b>0$ $\Rightarrow f''(x)>0$ up

$a<0$ + $b<0$ $\Rightarrow f''(x)<0$ down

concavity not conclusive

$12ax^2 + 2b \neq 0 \rightarrow$ no POI

33. If $f(x) = ax^4 + bx^2$ and $ab > 0$, then

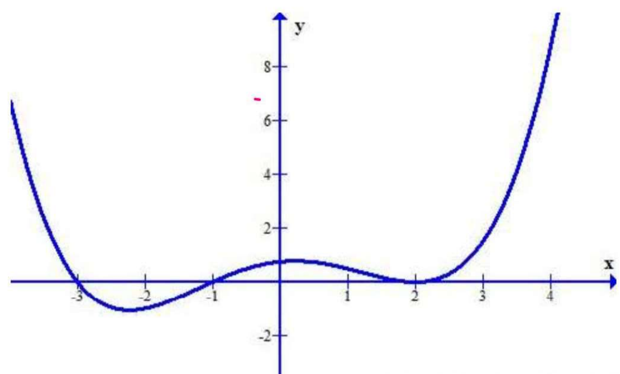
~~a)~~ The curve has no horizontal tangents $f'(0)=0$
~~b)~~ The curve is concave up for all x
~~c)~~ The curve is concave down for all x
$\boxed{\text{d)}}$ The curve has no inflection points
~~e)~~ None of the preceding are necessarily true

34. Let $f(x) = x^2$. Then $f(x)$ has a minimum value on which of the following intervals.

✓ I. $(-1,1)$ ~~II.~~ $(2,3)$ ✓ III. $(-5,-2]$

a. I only **b.** I & II Only $\boxed{\textbf{c.}}$ I & III Only **d.** I, II, & III **e.** None

Use the following graph to answer questions #34-37.



$f'(x)<0$ to $f'(x)>0$

35. The graph of $f'(x)$ is given above. Thus $f(x)$ has a local minimum at $x =$

a. -3 $\boxed{\textbf{b.}}$ -1 **c.** 0 **d.** 1 **e.** 2

$f'(x)>0$ to $f'(x)<0$

36. The graph of $f'(x)$ is given above. Thus $f(x)$ has a local maximum at $x =$

$\boxed{\textbf{a.}}$ -3 **b.** -1 **c.** 0 **d.** 1 **e.** 2

$f''(x)$ changes sign

37. The graph of $f''(x)$ is given above. Thus $f(x)$ has point(s) of inflection when $x =$

a. -1 & 1 **b.** -1 $\boxed{\textbf{c.}}$ -3 & -1 **d.** $-3, -1$ & 2 **e.** $-2, 0$ & 2

38. The graph of $f''(x)$ is given. The function $f(x)$ on the interval $(-1,2)$ is best described as

$f''(x)>0$

a. Increasing **b.** Decreasing $\boxed{\textbf{c.}}$ Concave Up **d.** Concave Down **e.** Positive