Integration by Parts is an integration technique that can be applied to many integrals like:

$$\int x\ln(x)dx \qquad \int x^2e^x dx \qquad \int e^x\sin(x)dx$$

$u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

Integration by Parts

If *u* and *v* are functions of *x* and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting ***dv*** be the most complicated portion of the integrand that fits a basic integration rule. Then ***u*** will be the remaining factor(s) of the integrand.
2. Try letting ***u*** the portion of the integrand whose derivative is a function simpler than *u*. Then ***dv*** will be the remaining factor(s) of the integrand.

Sample Problems: Evaluate the following integrals.
No, you do not have enough room to work on this sheet!

1. $\int xe^x dx$ $\quad u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = e^x$

$= xe^x - \int e^x dx$

$\boxed{xe^x - e^x + C}$

Check $\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x$

$= xe^x$ ✓

2. $\int x^2\ln(x)dx$ $\quad u = \ln x$ $\quad dv = x^2 dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^3}{3}$

$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}dx$

$\boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$

Check $\frac{d}{dx}\left(\frac{x^3}{3}\ln x - \frac{x^3}{9} + C\right)$

$= x^2\ln x + \frac{x^3}{3}\cdot\frac{1}{x} - \frac{x^2}{3}$

$= x^2\ln x$ ✓

3. $\int_0^1 \arcsin(x)dx$ $\quad u = \arcsin x$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $\quad v = x$

$\left[x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx\right]_0^1$

$w = 1-x^2$

$dw = -2x\,dx$

$dx = -\frac{1}{2x}dw$

$\left[x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{w}}dw\right]_0^1$

$\left[x\arcsin x + \frac{1}{2}\cdot 2\sqrt{w}\right]_0^1 = \left[x\arcsin x + \sqrt{1-x^2}\right]_0^1$

$= 1\arcsin 1 + \frac{1}{4}(0) - 0 - \sqrt{1} = \boxed{\frac{\pi}{2} - 1}$

4. $\int \sec^3(x)dx$ $\quad u = \sec x$ $\quad dv = \sec^2 x\,dx$

$du = \sec x\tan x\,dx$ $\quad v = \tan x$

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec x\tan^2 x\,dx$

$\tan^2 x = \sec^2 x - 1$

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C$

$\boxed{\int \sec^3 x\,dx = \frac{\sec x\tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C}$

5. $\int x^2 \sin(x)dx$ $u = x^2$, $du = 2x\,dx$, $dv = \sin x\,dx$, $v = -\cos x$

$= -x^2\cos x + \int 2x\cos x\,dx$

$u = 2x$, $du = 2dx$, $dv = \cos x\,dx$, $v = \sin x$

$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$

$\boxed{= -x^2\cos x + 2x\sin x + 2\cos x + C}$

Check: $\frac{d}{dx}(-x^2\cos x + 2x\sin x + 2\cos x + C)$

$= \cancel{-2x\cos x} + x^2\sin x + \cancel{2\sin x} + \cancel{2x\cos x} - \cancel{2\sin x}$

$= x^2\sin x$ ✓

6. $\int e^x \cos(x)dx$ $u = \cos x$, $du = -\sin x\,dx$, $dv = e^x dx$, $v = e^x$

$\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx$

$u = \sin x$, $du = \cos x\,dx$, $dv = e^x dx$, $v = e^x$

$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx + C$

$2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x + C$

$\boxed{\int e^x\cos x\,dx = \frac{e^x\cos x + e^x\sin x + C}{2}}$

7. $\int 4\arccos(x)dx$ $u = \arccos x$, $du = \frac{-1}{\sqrt{1-x^2}}dx$, $dv = 4dx$, $v = 4x$

$= 4x\arccos x + \int \frac{4x}{\sqrt{1-x^2}}dx$

$w = 1-x^2$, $dw = -2x\,dx$, $dx = \frac{dw}{-2x}$

$= 4x\arccos x - 2\int \frac{1}{\sqrt{w}}dw$

$= 4x\arccos x - 2\cdot 2\sqrt{w} + C = \boxed{4x\arccos x - 4\sqrt{1-x^2} + C}$

8. $\int \ln(3x)dx$ $u = \ln(3x)$, $du = \frac{1}{x}dx$, $dv = dx$, $v = x$

$= x\ln(3x) - \int x\cdot\frac{1}{x}dx$

$= \boxed{x\ln(3x) - x + C}$

9. $\int (3t+5)\cos\left(\frac{t}{4}\right)dt$ $u = 3t+5$, $du = 3dt$, $dv = \cos(\frac{t}{4})dt$, $v = 4\sin(\frac{t}{4})$

$= (3t+5)\cdot 4\sin(\frac{t}{4}) - \int 4\sin(\frac{t}{4})\cdot 3dt$

$= 4(3t+5)\sin(\frac{t}{4}) - 12\int \sin(\frac{t}{4})dt$

$= 4(3t+5)\sin(\frac{t}{4}) + 12\cdot 4\cos(\frac{t}{4})dt$

$= \boxed{4(3t+5)\sin(\frac{t}{4}) + 48\cos(\frac{t}{4}) + C}$

Check: $\frac{d}{dt}\left(4(3t+5)\sin(\frac{t}{4}) + 48\cos(\frac{t}{4}) + C\right)$

$= 12\sin(\frac{t}{4}) + 4(3t+5)\cdot\frac{1}{4}\cos(\frac{t}{4}) - \frac{48}{4}\sin(\frac{t}{4})$

$= (3t+5)\cos(\frac{t}{4})$ ✓

10. $\int x^2 \sin(10x)dx$ $u = x^2$, $du = 2x\,dx$, $dv = \sin(10x)dx$, $v = \frac{-1}{10}\cos(10x)$

$= \frac{-x^2}{10}\cos(10x) + \frac{2}{10}\int x\cos(10x)dx$

$u = x$, $du = dx$, $dv = \cos(10x)dx$, $v = \frac{1}{10}\sin(10x)$

$= \frac{-x^2}{10}\cos(10x) + \frac{1}{5}\left[\frac{x}{10}\sin(10x) - \frac{1}{10}\int \sin(10x)dx\right]$

$= \boxed{\frac{-x^2}{10}\cos(10x) + \frac{x}{50}\sin(10x) + \frac{\cos(10x)}{500} + C}$

11. $\int x\sqrt{x+1}\,dx$ $\quad u = x \quad du = dx \quad dv = \sqrt{x+1}\,dx \quad v = \frac{2}{3}(x+1)^{3/2}$

$$= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}\,dx$$

$$= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{5/2} + C$$

$$= \boxed{\frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C}$$

Check $\frac{d}{dx}\left(\frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C\right)$

$$= \frac{2}{3}(x+1)^{3/2} + x\sqrt{x+1} - \frac{2}{3}(x+1)^{3/2} = x\sqrt{x+1}\ \checkmark$$

12. $\int \ln(x)\,dx$ $\quad u = \ln x \quad du = \frac{1}{x}\,dx \quad dv = dx \quad v = x$

$$= x\ln x - \int x\cdot\frac{1}{x}\,dx$$

$$= \boxed{x\ln x - x + C}$$

Check: $\frac{d}{dx}(x\ln x - x + C)$

$$= \ln x + \frac{x}{x} - 1 = \ln x\ \checkmark$$

13. $\int x^5\sqrt{x^3+1}\,dx$ $\quad u = x^3+1 \quad du = 3x^2\,dx \quad dx = \frac{du}{3x^2}$

$$= \frac{1}{3}\int x^3\sqrt{u}\,du = \frac{1}{3}\int (u-1)\sqrt{u}\,du$$

$$= \frac{1}{3}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{3}\cdot\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$$

$$= \boxed{\frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C}$$

14. $\int x^4 e^{\frac{x}{2}}\,dx$

u		dv
x^4	+	$e^{\frac{x}{2}}$
$4x^3$	−	$2e^{\frac{x}{2}}$
$12x^2$	+	$4e^{\frac{x}{2}}$
$24x$	−	$8e^{\frac{x}{2}}$
24	+	$16e^{\frac{x}{2}}$
0		$32e^{\frac{x}{2}}$

$$= 2x^4e^{\frac{x}{2}} - 16x^3e^{\frac{x}{2}} + 96x^2e^{\frac{x}{2}} - 384xe^{\frac{x}{2}} + 768e^{\frac{x}{2}} + C$$

$$= \boxed{2e^{\frac{x}{2}}(x^4 - 8x^3 + 48x^2 - 192x + 384) + C}$$

15. $\int 2x\arctan(x)\,dx$ $\quad u = \arctan x \quad du = \frac{1}{x^2+1}\,dx \quad dv = 2x\,dx \quad v = x^2$

$$= x^2\arctan x - \int \frac{x^2+1-1}{x^2+1}\,dx$$

$$= x^2\arctan x - \int 1\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \boxed{x^2\arctan x - x + \arctan x + C}$$

Check: $\frac{d}{dx}(x^2\arctan x - x + \arctan x + C)$

$$= 2x\arctan x + \underbrace{\frac{x^2}{x^2+1} - \frac{x^2+1}{x^2+1} + \frac{1}{x^2+1}}_{=0}$$

$$= 2x\arctan x\ \checkmark$$

16. $\int \frac{\ln(x)}{x^5}\,dx$ $\quad u = \ln x \quad du = \frac{1}{x}\,dx \quad dv = x^{-5}\,dx \quad v = -\frac{x^{-4}}{4}$

$$= -\frac{\ln x}{4x^4} + \int \frac{x^{-5}}{4}\,dx$$

$$= \boxed{\frac{-\ln x}{4x^4} - \frac{1}{16x^4} + C}$$

Check: $\frac{d}{dx}\left(-\frac{\ln x}{4x^4} - \frac{1}{16x^4} + C\right)$

$$= -\frac{1}{4x^5} + \frac{\ln x}{x^5} + \frac{1}{4x^5} = \frac{\ln x}{x^5}\ \checkmark$$

17. $\int x^2 e^{2x}\,dx$

u	dv
x^2	e^{2x}
$2x$	$\frac{1}{2}e^{2x}$
2	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

$$= \boxed{\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C}$$

18. $\int \frac{\ln(x)}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

$$\int \frac{u}{\cancel{x}}\cancel{x}\,du$$

$$= \int u\,du = \frac{u^2}{2} + C$$

$$= \boxed{\frac{(\ln(x))^2}{2} + C}$$

Check: $\frac{d}{dx}\left(\frac{(\ln(x))^2}{2} + C\right)$

$= \ln x \cdot \frac{1}{x}$

$= \frac{\ln x}{x}$ ✓

19. $\int x^5(\cos x)\,dx$

u	dv
x^5	$\cos x$
$5x^4$	$\sin x$
$20x^3$	$-\cos x$
$60x^2$	$-\sin x$
$120x$	$\cos x$
120	$\sin x$
0	$-\cos x$

$$= \boxed{x^5\sin x + 5x^4\cos x - 20x^3\sin x - 60x^2\cos x + 120x\sin x + 120\cos x + C}$$

20. $\int \frac{2x}{e^x}\,dx$

u	dv
$2x$	e^{-x}
2	$-e^{-x}$
0	e^{-x}

$$= \boxed{-2xe^{-x} - 2e^{-x} + C}$$

21. $\int \frac{e^{\frac{1}{t}}}{t^2}\,dt$

$u = \frac{1}{t}$

$du = -\frac{1}{t^2}\,dt$

$dt = -t^2\,du$

$$-\int e^u\,du$$

$$= -e^u + C$$

$$= \boxed{-e^{\frac{1}{t}} + C}$$

22. $\int x^4(\ln x)\,dx$

$u = \ln x$ $\quad dv = x^4\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{x^5}{5}$

$$= \frac{x^5}{5}\ln x - \int \frac{x^4}{5}\,dx$$

$$= \boxed{\frac{x^5}{5}\ln x - \frac{x^5}{25} + C}$$

23. $\int e^{3x}\sin(2x)\,dx$

$u = e^{3x}$ $\quad dv = \sin(2x)dx$

$du = 3e^{3x}dx$ $\quad v = \dfrac{-\cos(2x)}{2}$

$$\int e^{3x}\sin(2x) = -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\int e^{3x}\cos(2x)\,dx$$

$u = e^{3x}$ $\quad dv = \cos(2x)dx$

$du = 3e^{3x}dx$ $\quad v = \frac{1}{2}\sin(2x)$

$$\int e^{3x}\sin(2x)dx = -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x) - \frac{9}{4}\int e^{3x}\sin(2x)dx$$

$$\frac{13}{4}\int e^{3x}\sin(2x)dx = -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x) + C$$

$$\int e^{3x}\sin(2x)dx = \boxed{\frac{-2}{13}e^{3x}\cos(2x) + \frac{3}{13}e^{3x}\sin(2x) + C}$$

24. $\int x\sin^{-1}\left(\frac{a}{x}\right)dx$

$u = \sin^{-1}\left(\frac{a}{x}\right)$ $\quad dv = x\,dx$

$du = \dfrac{1}{\sqrt{1-\frac{a^2}{x^2}}}\cdot\dfrac{-a}{x^2}dx$ $\quad v = \dfrac{x^2}{2}$

$$du = \frac{-a}{x\sqrt{x^2-a^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1}\left(\frac{a}{x}\right) + \frac{a}{2}\int\frac{x}{\sqrt{x^2-a^2}}dx$$

$w = x^2 - a^2$

$dw = 2x\,dx$

$dx = \dfrac{dw}{2x}$

$$= \frac{x^2}{2}\sin^{-1}\left(\frac{a}{x}\right) + \frac{a}{4}\int\frac{1}{\sqrt{w}}dw$$

$$= \frac{x^2}{2}\sin^{-1}\left(\frac{a}{x}\right) + \frac{a}{4}\cdot 2\sqrt{w} + C$$

$$= \boxed{\frac{x^2}{2}\sin^{-1}\left(\frac{a}{x}\right) + \frac{a}{2}\sqrt{x^2-a^2} + C}$$