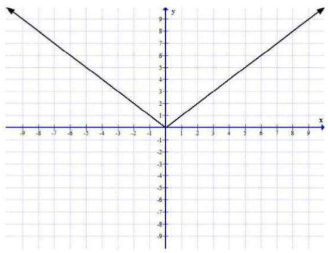
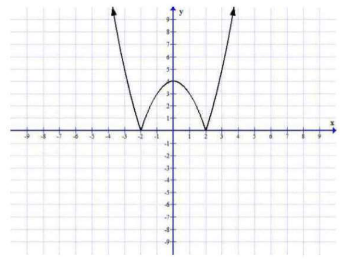
Do Now:

Sketch the graphs of $f(x)=|x|$ and $g(x)=|x^2-4|$. Write each as a piece-wise function.



$$f(x)=\begin{cases}-x, & x<0\\ 0, & x=0\\ x, & x>0\end{cases}$$



$$g(x)=\begin{cases}x^2-4, & |x|>2\\ -x^2+4, & -2<x<2\end{cases}$$

Classwork:

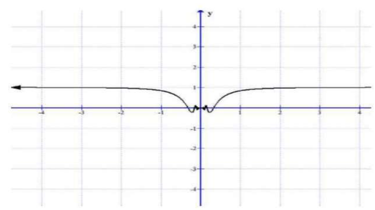
1. Evaluate $\lim\limits_{x\to 0}\dfrac{|x|}{x}$.

$\lim\limits_{x\to 0^-}\dfrac{-x}{x}=\lim\limits_{x\to 0^-}-1=-1$

$\lim\limits_{x\to 0^+}\dfrac{x}{x}=\lim\limits_{x\to 0^+}1=1$

$\lim\limits_{x\to 0}\dfrac{|x|}{x}$ DNE

2. What is $\lim\limits_{x\to\infty}x\sin\left(\dfrac{1}{x}\right)$? $\lim\limits_{x\to\infty}\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}=1$ $\lim\limits_{x\to\infty}\frac{1}{x}=0$



3. Evaluate $\lim\limits_{x\to 2^-}8\dfrac{|x^2-4|}{2-x}$. Show all work that leads to your final answer.

$8\lim\limits_{x\to 2^-}\dfrac{-x^2+4}{2-x}=8\lim\limits_{x\to 2^-}\dfrac{-(x-2)(x+2)}{-(x-2)}=8\lim\limits_{x\to 2^-}x+2=8(4)=32$

The Squeeze (Sandwich) Theorem

THEOREM 4 The Sandwich Theorem

If $g(x)\le f(x)\le h(x)$ for all $x\ne c$ in some interval about c, and

$$\lim_{x\to c}g(x)=\lim_{x\to c}h(x)=L,$$

then

$$\lim_{x\to c}f(x)=L.$$

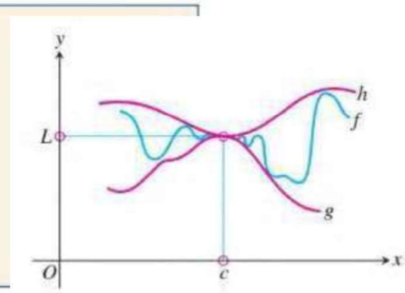
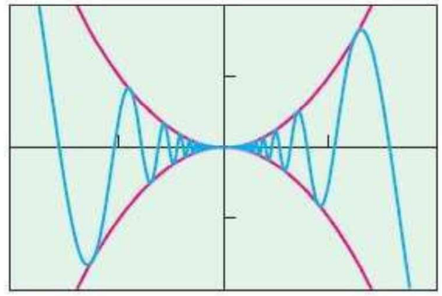


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

Sample Problems:

1. Show that the $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right) = 0$.



$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

$\lim\limits_{x \to 0} -x^2 = 0$

$\lim\limits_{x \to 0} x^2 = 0$

Since $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ and $\lim\limits_{x \to 0} x^2 = 0 = \lim\limits_{x \to 0} -x^2$, by the Squeeze Theorem $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

2. Evaluate $\lim\limits_{x \to 2}\left[5(x-2)^4 \cos\left(\dfrac{\pi}{x-2}\right)\right]$.

$-1 \le \cos\left(\frac{\pi}{x-2}\right) \le 1$

$-5(x-2)^4 \le 5(x-2)^4 \cos\left(\frac{\pi}{x-2}\right) \le 5(x-2)^4$

$\lim\limits_{x \to 2}\left(-5(x-2)^4\right) = 0 = \lim\limits_{x \to 2}\left(5(x-2)^4\right)$

Since $-5(x-2)^4 \le 5(x-2)^4 \cos\left(\frac{\pi}{x-2}\right) \le 5(x-2)^4$ and $\lim\limits_{x \to 2}\left(-5(x-2)^4\right) = 0 = \lim\limits_{x \to 2}\left(5(x-2)^4\right)$

By the Squeeze Theorem $\lim\limits_{x \to 2}\left[5(x-2)^4 \cos\left(\frac{\pi}{x-2}\right)\right] = 0$.

Practice: Show all work in your notebook for full credit.

1. Evaluate the following limits.

a. $\lim\limits_{x \to 2} \dfrac{x^2+4x-12}{x^2-4} = \lim\limits_{x \to 2} \dfrac{(x+6)(x-2)}{(x+2)(x-2)} =$

$= \lim\limits_{x \to 2} \dfrac{x+6}{x+2} = \dfrac{2+6}{2+2} = 2$

b. $\lim\limits_{x \to 3} \dfrac{x^3-27}{3-x} = \lim\limits_{x \to 3} \dfrac{(x-3)(x^2+3x+9)}{3-x}$

$= \lim\limits_{x \to 3} -x^2-3x-9$

$= -(3)^2 - 3(3) - 9$

$= -27$

c. $\lim\limits_{x \to 0} \dfrac{\sqrt{3+x}-\sqrt{3}}{x} \cdot \dfrac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}} =$

$\lim\limits_{x \to 0} \dfrac{3+x-3}{x(\sqrt{3+x}+\sqrt{3})} =$

$\lim\limits_{x \to 0} \dfrac{1}{\sqrt{3+x}+\sqrt{3}} = \dfrac{1}{2\sqrt{3}}$

d. $\lim\limits_{x \to 0} \dfrac{7(x-7)\left(\dfrac{1}{x-7}+\dfrac{1}{7}\right)}{7(x-7)(7x)} = \lim\limits_{x \to 0} \dfrac{7+x-7}{7x(7)(x-7)}$

$= \lim\limits_{x \to 0} \dfrac{x}{49x(x-7)}$

$= \lim\limits_{x \to 0} \dfrac{1}{49(x-7)}$

$= -\dfrac{1}{343}$

e. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 3x} = \lim\limits_{x\to 0} \dfrac{5\sin(5x)}{5x}\cdot\dfrac{3x}{3\sin(3x)}$

$= \dfrac{5}{3}\lim\limits_{x\to 0}\dfrac{\sin(5x)}{5x}\cdot\dfrac{3x}{\sin(3x)}$

$= \dfrac{5}{3}\cdot 1\cdot 1 = \boxed{\dfrac{5}{3}}$

f. $\lim\limits_{x\to 0}\dfrac{\sin x}{\tan x} = \lim\limits_{x\to 0} \sin x\left(\dfrac{\cos x}{\sin x}\right)$

$= \lim\limits_{x\to 0} \cos x = 1$

g. $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\sec x}{\sin x} = \dfrac{\sqrt{2}}{\frac{1}{\sqrt{2}}}$

$= \sqrt{2}\cdot\sqrt{2}$

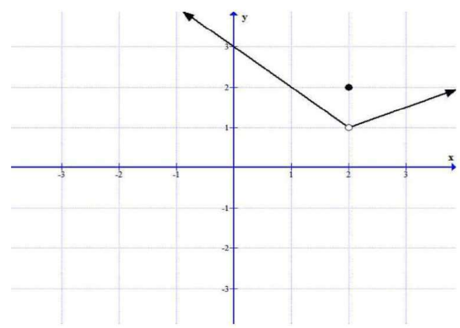
$= 2$

h. $\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)^2}{x^2} = \lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{1-\cos x}{x}\cdot 1-\cos x$

$= 1(0)(1-\cos(0))$

$= 0$

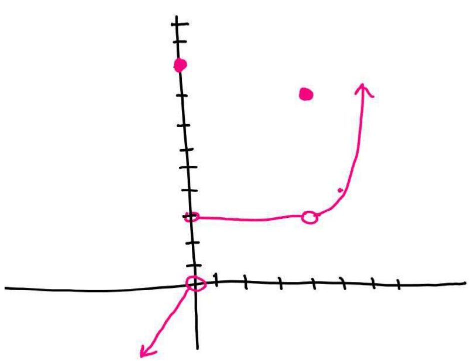
2. Given the piecewise defined function $f(x) = \begin{cases} 3-x, & x<2 \\ 2, & x=2 \\ \dfrac{x}{2}, & x>2 \end{cases}$, draw the graph of f and evaluate $\lim\limits_{x\to 2} f(x)$. Support your answer!



$\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = 3-2 = 1 \\ \lim\limits_{x\to 2^+} f(x) = \frac{2}{2} = 1 \end{array}\right\} \lim\limits_{x\to 2} f(x) = 1$

3. Draw and write a single function that satisfies all the following criteria.

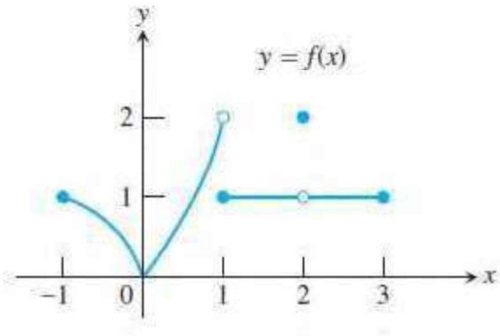
a. $f(4)=8$ b. $\lim\limits_{x\to 4} f(x) = 3$ c. $f(0)=9$ d. $\lim\limits_{x\to 0} f(x)$ Does Not Exist



$f(x) = \begin{cases} x, & x<0 \\ 9, & x=0 \\ 3, & 0<x<4 \\ 8, & x=4 \\ (x-4)^2+3, & x>4 \end{cases}$

Answer may (and should) vary

4. Find the following values.



a. $f(1) =$ 1

b. $f(0) =$ 0

c. $\lim_{x \to -1^+} f(x) =$ 1

d. $\lim_{x \to 1} f(x) =$ DNE

e. $\lim_{x \to 2} f(x) =$ 1

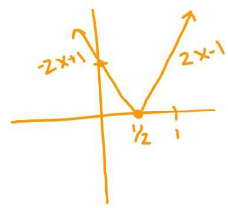
f. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) =$ $0 = \lim_{x \to 0} f(x)$

g. $\lim_{x \to 3^-} f(x) =$ 1

h. True or False: $\lim_{x \to c} f(x)$ exists for every c in $(-1, 1)$ True

5. Evaluate the $\lim_{x \to \frac{1}{2}^+} \frac{|2x-1|}{6x-3}$. Show all work that leads to your final answer.

$$\lim_{x \to \frac{1}{2}^+} \frac{2x-1}{3(2x-1)} = \lim_{x \to \frac{1}{2}^+} \frac{1}{3} = \frac{1}{3}$$



6. Find $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ if $f(x) = x^2 - 8x$.

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 8(x+\Delta x) - x^2 + 8x}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 8x - 8\Delta x - x^2 + 8x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 - 8(\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x - 8 = 2x - 8$$