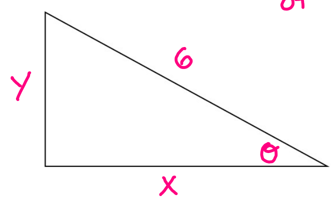
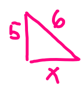
1. Brian is standing on a 6-meter ladder that is leaning against a wall when Roger begins to pull the bottom of the ladder out away from the wall. The *bottom* of the ladder is pulled at a constant rate of 0.5 m/s.

a. How fast is the top of the ladder moving when it is 5 meters up the wall?

b. How fast is the angle formed between the ladder and ground changing at this instant?

$\frac{dx}{dt} = \frac{1}{2}$ m/s





$x^2 + 25 = 36$

$x = \sqrt{11}$

A. $\frac{d}{dt}\left(x^2 + y^2 = 6^2\right)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$

$\frac{dy}{dt} = -\frac{\sqrt{11}}{5}\left(\frac{1}{2}\right) = \boxed{-\frac{\sqrt{11}}{10} \text{ m/sec}}$

B. $\frac{d}{dt}\left(\cos\theta = \frac{x}{6}\right)$

$-\sin\theta \frac{d\theta}{dt} = \frac{1}{6}\frac{dx}{dt}$

$\frac{d\theta}{dt} = -\frac{1}{6}\csc\theta \frac{dx}{dt}$

$\frac{d\theta}{dt} = -\frac{1}{6}\left(\frac{6}{5}\right)\left(\frac{1}{2}\right) = \boxed{\frac{-1}{10} \text{ rad/sec}}$

2. The radius of a sphere is increasing at a constant rate of 2 in/min. Find the rate of change of the volume of the sphere when the radius is 6 inches and 24 inches.

$\frac{dr}{dt} = 2$ in/min

$\frac{d}{dt}\left(V = \frac{4\pi}{3}r^3\right)$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\left.\frac{dV}{dt}\right|_{r=6} = 4\pi(6)^2(2) = \boxed{288\pi \text{ in}^3/\text{min}}$

$\left.\frac{dV}{dt}\right|_{r=24} = 4\pi(24)^2(2) = \boxed{4{,}608\pi \text{ in}^3/\text{min}}$

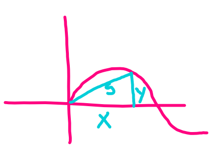
3. A point is moving along the graph of $y = \sqrt{x}$ in a manner such that $\frac{dx}{dt} = 3$ cm/sec. Find $\frac{dy}{dt}$ when $x = 4$.

$\frac{d}{dt}\left(y = \sqrt{x}\right)$

$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt}$

$\left.\frac{dy}{dt}\right|_{x=4} = \frac{1}{2\sqrt{4}}(3) = \boxed{\frac{3}{4} \text{ cm/sec}}$

4. Find the rate of change of the distance between the origin and a point moving on the graph of $y = \sin x$ if the rate of change of the x-coordinate with respect to time at $x = \pi$ is 2 cm/sec.



$x^2 + y^2 = s^2$

$\frac{d}{dt}\left(x^2 + \sin^2 x = s^2\right)$

$2x\frac{dx}{dt} + 2\sin x\cos x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + \sin x\cos x\frac{dx}{dt}}{s}$

$\left.\frac{ds}{dt}\right|_{x=\pi} = \frac{2\pi + \sin\pi\cos\pi\cdot 2}{\pi}$

$\boxed{\left.\frac{ds}{dt}\right|_{x=\pi} = 2 \text{ cm/sec}}$

$\frac{dx}{dt} = 2$ cm/sec

$\underbrace{x = \pi \rightarrow y = \sin\pi = 0}$

$\pi^2 + 0^2 = s^2$

$s = \pi$

5. All edges of a cube are expanding at a rate of 3 cm/sec. How fast is the volume changing when each edge of the cube is 10 cm? How fast is the surface area of the cube changing at this instant?

$\frac{ds}{dt} = 3$ cm/sec

$V = s^3$

$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$

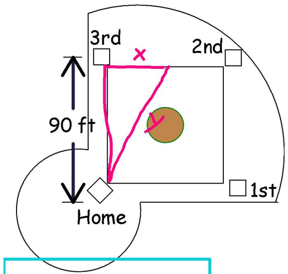
$\frac{dV}{dt} = 3(10)^2(3) = \boxed{900 \text{ cm}^3/\text{sec}}$

$A = 6s^2$

$\frac{dA}{dt} = 12s\frac{ds}{dt}$

$\frac{dA}{dt} = 12(10)(3) = \boxed{360 \text{ cm}^2/\text{sec}}$

6. A baseball diamond has the shape of a square with sides 90 feet long *(see figure)*. A player running from second base to third base at a speed of 28 feet per second is 30 feet from third base. At what rate is the player's distance, *s*, from home plate changing?



$\frac{dx}{dt} = -28 \text{ ft/sec}$ Find $\frac{ds}{dt}$ when $x = 30 \rightarrow y =$

$90^2 + x^2 = y^2$

$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$

$90^2 + 30^2 = y^2$

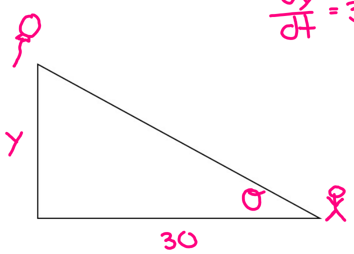
$y^2 = 9000$

$y = 30\sqrt{10}$

$\left.\frac{dy}{dt}\right|_{x=30} = \frac{30}{3\sqrt{10}} \cdot (-28)$

$= -\frac{280}{\sqrt{10}} = \frac{-280\sqrt{10}}{10} = \boxed{-28\sqrt{10} \text{ ft/sec}}$

7. A balloon rises at a rate of 30 meters per second from a point on the ground 30 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground.



$\frac{dy}{dt} = 30 \text{ m/sec}$ Find $\frac{d\theta}{dt}$ when $y = 30$

$\tan\theta = \frac{y}{30}$

$\frac{d}{dt}\left(\tan\theta = \frac{1}{30}y\right)$

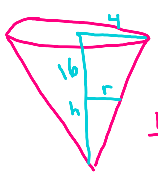
$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{\cos^2\theta}{30}\frac{dy}{dt}$

$\left.\frac{d\theta}{dt}\right|_{y=30} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{30} \cdot 30$

$= \boxed{\frac{1}{2} \text{ rad/sec}}$

8. Water is flowing into a cone at a rate of $2\text{cm}^3/\text{min}$. The cone has a height of 16 cm and a radius of 4 cm. How fast is the water level rising when it is **(a)** 5cm deep and **(b)** 10cm deep?



$\frac{dV}{dt} = 2 \text{ cm}^3/\text{min}$ Find $\frac{dh}{dt}$ when $h = 5$ & $h = 10$

$\frac{16}{h} = \frac{4}{r}$

$16r = 4h$

$r = \frac{4h}{16} = \frac{h}{4}$

$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{48}$

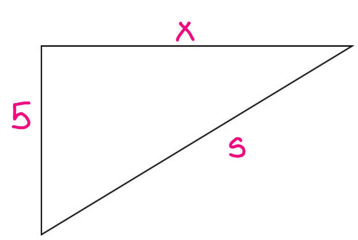
$\frac{dV}{dt} = \frac{3\pi h^2 \frac{dh}{dt}}{48} = \frac{\pi h^2 \frac{dh}{dt}}{16}$

$\frac{16\frac{dV}{dt}}{\pi h^2} = \frac{dh}{dt}$

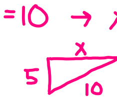
$\left.\frac{dh}{dt}\right|_{h=5} = \frac{16(2)}{25\pi} = \boxed{\frac{32}{25\pi} \text{ cm/min}}$

$\left.\frac{dh}{dt}\right|_{h=10} = \frac{16(2)}{10^2\pi} = \frac{32}{100\pi} = \boxed{\frac{8}{25\pi} \text{ cm/min}}$

9. An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna. When the plane is 10 miles away from the radar, the radar detects that the distance, *s*, is changing at a rate of 240 miles per hour. What is the speed of the plane?



$\frac{ds}{dt} = 240 \text{ m/hr}$ Find $\frac{dx}{dt}$ when $s = 10 \rightarrow x = 5\sqrt{3}$



$5^2 + x^2 = 100$

$x^2 = 75$

$x = 5\sqrt{3}$

$\frac{d}{dt}\left(x^2 + 5^2 = s^2\right)$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$\frac{dx}{dt} = \frac{s\frac{ds}{dt}}{x}$

$\left.\frac{dx}{dt}\right|_{s=10} = \frac{10(240)}{5\sqrt{3}} = \frac{480}{\sqrt{3}} = \frac{480\sqrt{3}}{3} = \boxed{160\sqrt{3} \text{ m/hr}}$

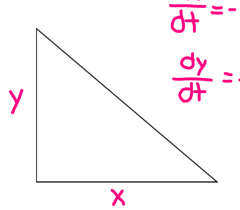
10. An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other. One plane is 150 miles from the point moving at 450 miles per hour. The other plane is 200 miles from the point moving at 600 miles per hour.

a. At what rate is the distance between the planes decreasing?

b. How much time does the air traffic controller have to get one of the planes on a different flight path?

A $\frac{dx}{dt} = -450 \text{ m/h}$

$\frac{dy}{dt} = -600 \text{ m/h}$



Find $\frac{ds}{dt}$ when $x = 150$ & $y = 200 \rightarrow s = 250$

$$s^2 = x^2 + y^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s}$$

$$\left.\frac{ds}{dt}\right|_{x=150} = \frac{150(-450) + 200(-600)}{250} \boxed{= -750 \text{ m/hr.}}$$

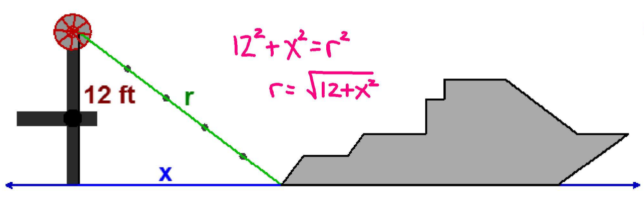
B. $d = rt$

$250 = 750t$

$\frac{1}{3} = t$

$\boxed{\text{20 minutes}}$

11. A boat is being pulled into a dock by means of a winch 12 feet above the water. The winch is pulling in the rope at a constant rate of 4 feet per second. Determine the speed at which the boat is approaching the dock when there is a total of 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?



$12^2 + x^2 = r^2$

$r = \sqrt{12 + x^2}$

A. $\frac{dr}{dt} = -4 \text{ ft/sec}$ Find $\frac{dx}{dt}$ when $r = 13 \rightarrow x = 5$

$$12^2 + x^2 = r^2$$

$$2x\frac{dx}{dt} = 2r\frac{dr}{dt}$$

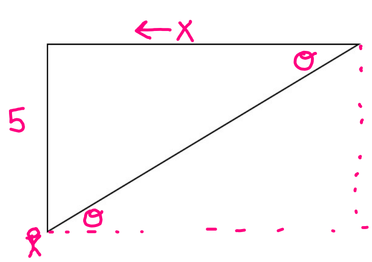
$$\frac{dx}{dt} = \frac{r}{x}\frac{dr}{dt} = \frac{-4r}{x}$$

$$\left.\frac{dx}{dt}\right|_{r=13} = \frac{13}{5}(-4) \boxed{= \frac{-52}{5} \text{ ft/sec}}$$

B. $\lim_{x \to 0} -\frac{4r}{x} = \lim_{x \to 0} \frac{-4\sqrt{12 + x^2}}{x} = -\infty$ shows moving towards dock

$\boxed{\text{Speed is increasing}}$

12. An airplane flies at an altitude of 5 miles toward a point directly over an observer. The speed of the plane is 600 miles per hour. Find the rate at which the angle of elevation from the observer, θ, is changing when the angle is $\theta = 30°$



$\frac{dx}{dt} = -600 \text{ m/h}$

$\theta = 30°$

$\tan 30 = \frac{5}{x}$

$\frac{1}{\sqrt{3}} = \frac{5}{x}$

$x = 5\sqrt{3}$

$$\tan\theta = \frac{5}{x}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-5}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{5}{x^2}\cos^2\theta \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-5}{(5\sqrt{3})^2}\cos^2(30)(-600)$$

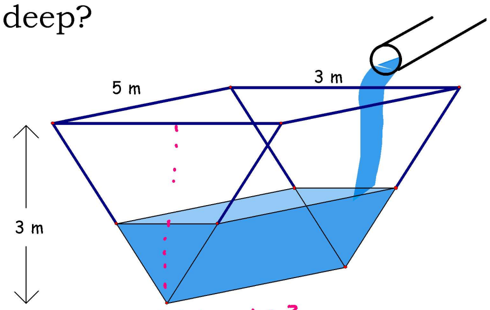
$$= \frac{-5}{75}\left(\frac{\sqrt{3}}{2}\right)^2(-600)$$

$$= \frac{-1}{15}\left(\frac{3}{4}\right)(-600)$$

$$\boxed{= 30 \text{ rad/hr}}$$

13. The cross-section of a 5-meter trough is an isosceles triangle with a 3-meter base and an altitude of 3 meters. Water is running into the trough at a rate of 1 cubic meter per minute. How fast is the water level rising when the water is 1 meter deep?

$\frac{dV}{dt} = 1\ m^3/min$ Find $\frac{dh}{dt}$ when $h = 1m$



$A = \frac{1}{2}bh = \frac{1}{2}h^2$

$V = 5A = \frac{5}{2}h^2$

$\frac{dV}{dt} = 5h\frac{dh}{dt}$

$\frac{dV}{dt}\left(\frac{1}{5h}\right) = \frac{dh}{dt}$

$\left.\frac{dh}{dt}\right|_{h=1} = 1\left(\frac{1}{5(1)}\right) = \boxed{\frac{1}{5}\ m/min}$

14. Water runs into a conical tank at the rate of 9 ft³/min. The tank stands vertex down and has a height of 10 feet and a base radius of 5 feet. How fast is the water level rising when the water is 6 feet deep?

$\frac{dV}{dt} = 9\ ft^3/min$ Find $\frac{dh}{dt}$ when $h = 6$

$\frac{h}{10} = \frac{r}{5}$

$10r = 5h$

$r = \frac{1}{2}h$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

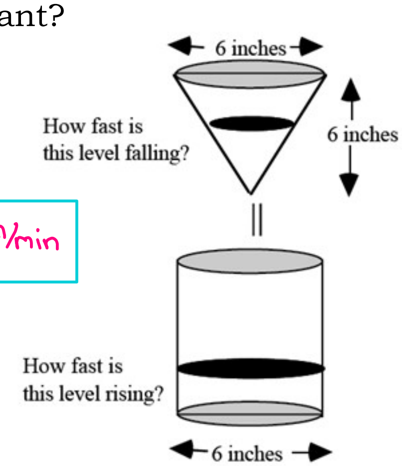
$\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{\pi h^2}$

$\left.\frac{dh}{dt}\right|_{h=6} = \frac{4(9)}{\pi(6^2)} = \boxed{\frac{1}{\pi}\ ft/min}$

15. Coffee is draining from a conical filter into a cylindrical teapot at the rate of 10 in³/min.

a. How fast is the level in the pot rising when the coffee in the filter is 5 inches deep?

b. How fast is the level in the conical filter falling at the same instant?



A. $\frac{dV}{dt} = -10\ in^3/min$

$\frac{6}{c} = \frac{3}{r}$

$\frac{c}{2} = r$

$V = \frac{\pi r^2 c}{3} = \frac{c^3\pi}{12}$

$\frac{dV}{dt} = \frac{3c^2\frac{dc}{dt}\pi}{12} = \frac{c^2\frac{dc}{dt}\pi}{4}$

$\frac{4\frac{dV}{dt}}{c^2\pi} = \frac{dc}{dt} \rightarrow \left.\frac{dc}{dt}\right|_{c=5} = \frac{-40}{25\pi} = \boxed{\frac{-8}{5\pi}\ in/min}$

B. $\frac{dV}{dt} = 10\ in^3/min$

$r = 3 \rightarrow$ constant

$V = \pi r^2 h$

$V = 9\pi h$

$\frac{dV}{dt} = 9\pi\frac{dh}{dt}$

$\frac{1}{9\pi}\cdot\frac{dV}{dt} = \frac{dh}{dt}$

$\frac{dh}{dt} = \boxed{\frac{10}{9\pi}\ in/min}$

16. A particle is moving along the curve whose equation is $\frac{xy^3}{1+y^2} = \frac{8}{5}$. Assume the x-coordinate is increasing at the rate of 6 units/sec when the particle is at the point (1,2). At what rate is the y-coordinate of the point changing at that instant? Is it rising or falling?

$\frac{dx}{dt} = 6$ units/sec at (1,2)

$$\frac{(1+y^2)\left(\frac{dx}{dt}y^3 + x3y^2\frac{dy}{dt}\right) - xy^3\left(2y\frac{dy}{dt}\right)}{(1+y^2)^2} = 0$$

$$\frac{(1+4)\left(6(2)^3 + 3(1)(2)^2\frac{dy}{dt}\right) - 1(2)^3\left(2(2)\frac{dy}{dt}\right)}{(1+2^2)^2} = 0$$

$5\left(48 + 12\frac{dy}{dt}\right) - 32\frac{dy}{dt} = 0$

$\rightarrow 240 + 60\frac{dy}{dt} - 32\frac{dy}{dt} = 0$

$28\frac{dy}{dt} = -240$

$\boxed{\frac{dy}{dt} = \frac{-60}{7}\ units/sec}$

17. Sand falls from a conveyor belt at a rate of 30 ft³/min onto the top of a conical pile. The height of the pile is always $\frac{3}{8}$ of the base diameter. How fast is the height changing when the pile is 12 feet high?

$\frac{dV}{dt} = 30 \text{ ft}^3/\text{min}$

$h = \frac{3}{8}d$

$d = \frac{8}{3}h$

$r = \frac{8}{6}h = \frac{4}{3}h$

$V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(\frac{4}{3}h\right)^2 h = \frac{16\pi h^3}{27}$

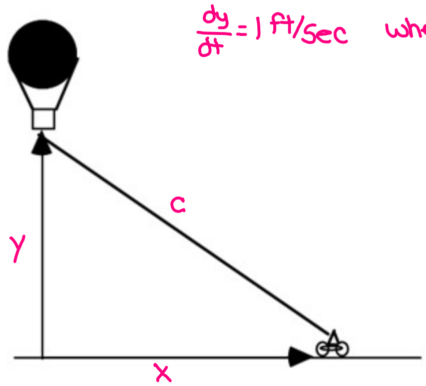
$\frac{dV}{dt} = \frac{16\pi}{9}h^2\frac{dh}{dt}$

$\frac{9}{16\pi h^2}\left(\frac{dV}{dt}\right) = \frac{dh}{dt}$

$\left.\frac{dh}{dt}\right|_{h=12} = \frac{9}{16\pi(12)^2}\cdot 30 = \frac{270}{2304\pi} = \boxed{\frac{15}{128\pi} \text{ ft/min}}$

18. A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 feet above the ground, a bicycle passes under it going 17 ft/sec. How fast is the distance between the bicycle and balloon increasing 3 seconds later?

$\frac{dy}{dt} = 1$ ft/sec when $y=65$, $x=0$ & $\frac{dx}{dt} = 17$ ft/sec $\Rightarrow$ 3 seconds later $y=68$, $x=51$, $c=85$



$x^2+y^2=c^2$

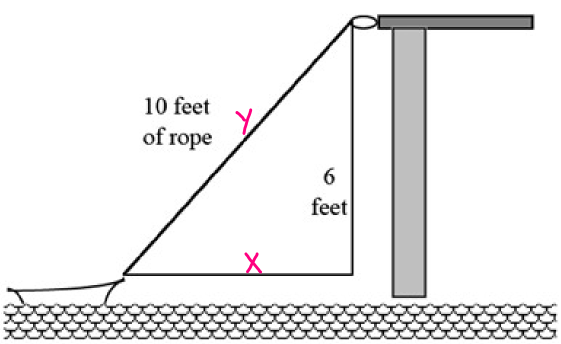
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2c\frac{dc}{dt}$

$51(17)+68(1) = 85\frac{dc}{dt}$

$\boxed{\frac{dc}{dt} = 11 \text{ ft/sec}}$

19. A rowboat is pulled toward a dock from the bow through a ring on the dock 6 feet above the bow. If the rowboat is hauled in at 2 ft/sec, how fast is the boat approaching the dock when 10 feet of rope are out?

$\frac{dy}{dt} = -2$ ft/sec Find $\frac{dx}{dt}$ when $y=10 \rightarrow x=8$



$x^2+6^2=y^2$

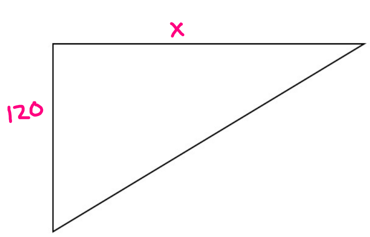
$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$\frac{dx}{dt} = \frac{y\frac{dy}{dt}}{x}$

$\left.\frac{dx}{dt}\right|_{y=10} = \frac{10(-2)}{8} = \boxed{\frac{-5}{2} \text{ ft/sec}}$

20. A boy flies a kite which is 120 feet directly above his hand. If the wind carries the kite horizontally at the rate of 30 ft/min, at what rate is the string being pulled out when the length of the string is 150 feet?

$\frac{dx}{dt} = 30$ ft/min Find $\frac{ds}{dt}$ when $s=150 \rightarrow x=90$



$x^2+120^2=s^2$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{x\frac{dx}{dt}}{s}$

$\left.\frac{ds}{dt}\right|_{s=150} = \frac{90(30)}{150} = \boxed{18 \text{ ft/sec}}$