They 1617

AP Calculus AB

Exponential and Logarithmic Derivatives
Sections 5.1, 5.4, & 5.5

Do Now: Review of Properties of Exponential and Logarithmic Functions

1. Write $\log_5 8$ in terms of natural logarithms. $\log_5 8 = \frac{\ln 8}{\ln 5}$

2. Write 7^x as a power of e. $7^x = e^{\ln 7^x} = e^{x\ln 7}$

3. Simplify the expressions using properties of exponents and logarithms.

   a. $\ln\left(e^{\tan x}\right) = \tan x$

   b. $\log_2 8^{x-5} = \log_2 2^{3(x-5)} = 3(x-5)$

   c. $3\ln(x) - \ln(3x) + \ln(12x^2) = \ln \frac{x^3(12x^2)}{3x} = \ln 4x^4$

   d. $\ln(x^2-4) - \ln(x+2) = \ln(x-2)$

4. Expand $\ln\dfrac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}$. $2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$

5. Solve the following equations and leave the answers in exact values.

   a. $3^x = 19$

   $x = \log_3 19$

   b. $\log_2 x = -4$

   $x = 2^{-4} = \frac{1}{16}$

   c. $3^{x+1} = \dfrac{1}{27}$

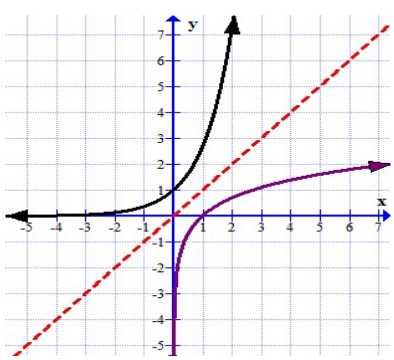
   $3^{x+1} = 3^{-3}$

   $\therefore x+1 = -3$

   $x = -4$

6. Sketch $y = e^x$ and $y = \ln x$. State the domain and range of each function.

   Discuss the relationship between the two graphs. How is this relationship shown graphically?



$y = e^x$

D: $(-\infty, \infty)$

R: $(0, \infty)$

$y = \ln(x)$

D: $(0, \infty)$

R: $(-\infty, \infty)$

$y = e^x$ and $y = \ln(x)$ are inverse functions. They are reflected over the line $y = x$.

Class Work:

1. Use the fact that $\lim\limits_{h\to 0}\dfrac{e^h - 1}{h} = 1$ to find $\dfrac{d}{dx}\left(e^x\right)$.

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right) = e^x(1) = e^x$$

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

2. Find the derivative of $\frac{d}{dx}\left(a^x\right)$. $a^x = e^{\ln a^x} = e^{x\ln a}$

$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$

$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\cdot \ln a = e^{\ln a^x}\cdot \ln a = a^x \ln a$

3. Find the derivative of $y = \ln(x)$. $\Rightarrow$ $e^y = x$

$\frac{d}{dx}(e^y = x)$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

4. Find the derivative of $\frac{d}{dx}(\log_a x)$. $\log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a}(\ln x)$

$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}}$

$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{1}{\ln a}\ln x\right) = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{x\ln a}$

5. Find $\frac{dy}{dx}$ if $y = e^{x+x^2}$.

$y' = e^{x+x^2}(1+2x) = (2x+1)e^{x+x^2}$

6. At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21?

$y' = 2^t \ln 2$

$2^t \ln 2 = 21$

$2^t = \frac{21}{\ln 2}$

$\log_2\left(\frac{21}{\ln 2}\right) = t$

$y = 2^{\log_2\left(\frac{21}{\ln 2}\right)} - 3 = \frac{21}{\ln 2} - 3$

$\boxed{\left(\log_2\left(\frac{21}{\ln 2}\right), \frac{21}{\ln 2} - 3\right)}$

7. A line with $\underbrace{\text{slope } m \text{ passes through the origin}}_{y=mx}$ and is tangent to the graph of $y = \ln x$. What is the value of m?

$y' = \frac{1}{x}$

$m = \frac{1}{x}$

$y = \frac{1}{x}(x) = 1 \Rightarrow 1 = \ln x$

$x = e$

$\boxed{m = \frac{1}{e}}$

8. Find $\frac{dy}{dx}$ if $y = x^x$ where $x > 0$.

$\ln y = x \ln x$

$\frac{1}{y} y' = \ln x + x\left(\frac{1}{x}\right)$

$y' = y(\ln x + 1)$

$\boxed{y' = x^x(\ln x + 1)}$

9. Find the derivative of the following functions:

a. $y = e^{\frac{2x}{3}}$

$y' = e^{\frac{2x}{3}} \cdot \frac{2}{3}$

$\boxed{y' = \frac{2}{3} e^{\frac{2x}{3}}}$

b. $y = 9^{-x}$

$y' = 9^{-x} \ln 9 (-1)$

$\boxed{y' = -9^{-x} \ln 9}$

c. $y = \ln(8x)$

$y' = \frac{1}{8x} \cdot 8$

$\boxed{y' = \frac{1}{x}}$

d. $y = \log\sqrt{x^2 - 1}$

$y = \frac{1}{2}\log(x^2 - 1)$

$y' = \frac{1}{2\ln 10\,(x^2-1)} \cdot 2x$

$\boxed{y' = \frac{x}{\ln 10\,(x^2 - 1)}}$

10. The spread of a flu in a certain school is modeled by the equation $P(t) = \frac{100}{1 + e^{3-t}}$, where $P(t)$ is the total number of students infected t days after the flu was first noticed. Many of them may already be well again at time *t*.

a. Estimate the initial numbers of students infected with the flu. $P(0) = \frac{100}{1 + e^3}$ $\boxed{\approx 5 \text{ students}}$

b. How fast is the flu spreading after 3 days?

c. When will the flu spread at its maximum rate? What is this rate?

b $P(t) = 100(1 + e^{3-t})^{-1}$

$P'(t) = -100(1 + e^{3-t})^{-2} e^{3-t}(-1)$

$= \frac{100 e^{3-t}}{(1 + e^{3-t})^2}$

$P'(3) \approx \frac{100 e^0}{(1 + e^0)^2} = \frac{100}{4}$ $\boxed{= 25 \text{ students/day}}$

C. max of $P'(t) \Rightarrow$

$\boxed{\begin{array}{l} 3 \text{ days} \\ 25 \text{ students/day} \end{array}}$

11. Show that for a differentiable function u where $u < 0$, $\frac{d}{dx}[\ln|u|] = \frac{u'}{u}$. Use this to find the derivative of $y = \ln|\csc x + \cot x|$.

$\frac{d}{dx}(\ln|u|) = \frac{d}{dx}(\ln(-u)) = \frac{1}{-u} \cdot -u' = \frac{u'}{u}$

$\frac{d}{dx}(\ln|\csc x + \cot x|) = \frac{1}{\csc x + \cot x} \cdot -\csc x \cot x - \csc^2 x = \frac{-\csc x(\cot x + \csc x)}{\csc x + \cot x}$ $\boxed{= -\csc x}$

12. Let $y = \dfrac{e^x + e^{-x}}{2}$.

a. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{1}{2}\left(e^x + e^{-x}(-1)\right) = \boxed{\frac{1}{2}\left(e^x - e^{-x}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(e^x - e^{-x}(-1)\right) = \boxed{\frac{1}{2}\left(e^x + e^{-x}\right)}$$

b. Find an equation of the line tangent to the curve at $x = 1$.

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2}\left(e^1 - e^{-1}\right) = \frac{1}{2}\left(e - \frac{1}{e}\right) = \frac{e^2-1}{2e}$$

$$y|_{x=1} = \frac{e + \frac{1}{e}}{2} = \frac{e^2+1}{2e}$$

$$\boxed{y - \frac{e^2+1}{2e} = \frac{e^2-1}{2e}(x-1)}$$

c. Find an equation of the line normal to the curve at $x = 1$.

$$\boxed{y - \frac{e^2+1}{2e} = \frac{-2e}{e^2-1}(x-1)}$$

d. Find any points where the tangent line is horizontal.

$$\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right) = 0$$

$$e^x - e^{-x} = 0$$

$$e^x = e^{-x} \Rightarrow x = 0$$

$$\boxed{(0,1)}$$

13. Find an equation for a line that is normal to the graph of $y = xe^x$ and goes through the origin.

$$y' = e^x + xe^x$$

$$y'|_{(0,0)} = e^0 + 0e^0 = 1$$

$$m_T = 1$$

$$m_n = -1$$

$$y - 0 = -1(x-0)$$

$$\boxed{y = -x}$$

14. At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to the line $y = 5x - 1$?

$$y' = 3^x \ln 3 = 5$$

$$3^x = \frac{5}{\ln 3}$$

$$x = \log_3 \frac{5}{\ln 3}$$

$$y = 3^{\log_3 \frac{5}{\ln 3}} + 1 = \frac{5}{\ln 3} + 1$$

$$\boxed{\left(\log_3 \frac{5}{\ln 3}, \frac{5}{\ln 3} + 1\right)}$$

15. A line with $\underbrace{\text{slope } m \text{ passes through the origin}}_{y = mx}$ and is tangent to $y = \ln(2x)$. What is the value of m?

$$y' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$$

$$m = \frac{1}{x}$$

$$y = \frac{1}{x}(x) = 1 \Rightarrow 1 = \ln(2x)$$

$$e = 2x$$

$$x = \frac{e}{2}$$

$$m = \frac{1}{x} = \frac{1}{\frac{e}{2}} = \frac{2}{e}$$

$$\boxed{m = \frac{2}{e}}$$

16. Find $\frac{dy}{dx}$.

a. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$

$\ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1)$

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{2x}{2(x^2+1)} - \frac{2}{3(x+1)}$

$$\boxed{\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)}$$

b. $y = x^2e^x - xe^x$

$y' = 2xe^x + x^2e^x - e^x - xe^x = \boxed{x^2e^x + xe^x - e^x}$

c. $y = (\sin x)^x,\ 0 < x < \frac{\pi}{2}$

$\ln y = x\ln(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + \frac{x}{\sin x}\cdot \cos x$

$$\boxed{\frac{dy}{dx} = (\sin x)^x\left(\ln(\sin x) + x\cot x\right)}$$

d. $y = (\ln x)^2$

$y' = 2\ln x\left(\frac{1}{x}\right) = \boxed{\frac{2\ln x}{x}}$

e. $y = \log_5\sqrt{x} = \frac{1}{2}\log_5 x$

$y' = \frac{1}{2\ln 5\, x} = \boxed{\frac{1}{x\ln 25}}$

f. $y = x^{\ln x}$

$\ln y = \ln x \ln x$

$\frac{d}{dx}\left(\ln y = (\ln x)^2\right)$

$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x}$

$\frac{dy}{dx} = \frac{2y\ln x}{x} = \boxed{\frac{2x^{\ln x}\ln x}{x}}$

g. $y = 3^{\cot x}$

$y' = 3^{\cot x}\ln 3\,(-\csc^2 x)$

$$\boxed{y' = 3^{\cot x}\ln\left(\tfrac{1}{3}\right)\csc^2 x}$$

h. $y = x^{\pi}$ *Power Rule*

$$\boxed{y' = \pi x^{\pi-1}}$$

i. $y = \ln\left(\frac{10}{x}\right) = \ln 10 - \ln x$

$$\boxed{y' = -\frac{1}{x}}$$

j. $y = 9^{-x^2}$

$y' = 9^{-x^2} \ln 9 \cdot (-2x)$

$y' = -2x \ln 9 \cdot 9^{-x^2}$

$\boxed{y' = x \ln\left(\frac{1}{81}\right) \cdot 9^{-x^2}}$

k. $y = \log_3 (1 + x \ln 3)$

$y' = \frac{1}{(1+x\ln 3)\ln 3} \cdot \ln 3 = \frac{1}{1 + x\ln 3}$

$\boxed{y' = \frac{1}{1+x\ln 3}}$

l. $y = \ln 2 \bullet \log_2 x = \ln 2\left(\frac{\ln x}{\ln 2}\right) = \ln x$

$\boxed{y' = \frac{1}{x}}$