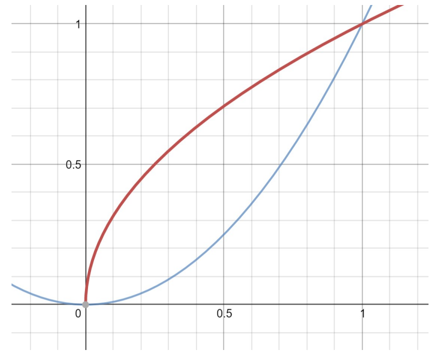
**Do Now:**




1. Sketch and shade the area bounded by the curves $y = \sqrt{x}$ and $y = x^2$. No calculators allowed!
2. Find the intersection of the two curves algebraically.

$(\sqrt{x})^2 = (x^2)^4$

$x = x^4$

$x - x^4 = 0$

$x(1-x^3) = 0$

$x(1-x)(\underbrace{1+x+x^2}_{\text{imaginary roots}}) = 0$

$x=0 \quad x=1$

3. Find the area between the two curves algebraically.

$\int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx = \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\Big]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

**Class Notes:**

The **area between two curves** $f(x)$ and $g(x)$, where $f(x) > g(x)$, that is bounded by the vertical lines $x = a$ and $x = b$ is equal to $\int_a^b f(x) - g(x)\,dx$.

**Sample Problems:**

**1.** Determine the area of the region bounded by $y = xe^{-x^2}$, $y = x+1$, $x = 2$, and the $y$-axis.

$\int_0^2 x+1-xe^{-x^2}\,dx = \frac{1}{2}x^2 + x + \frac{1}{2}e^{-x^2}\Big]_0^2 =$

$2+2+\frac{1}{2}e^{-4} - 0 - 0 - \frac{1}{2} =$

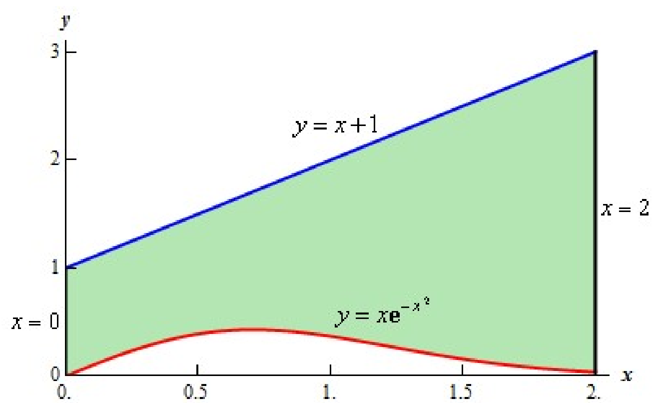
$\boxed{\frac{7}{2} + \frac{1}{2e^4}}$

$\int -xe^{-x^2}\,dx = \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^{-x^2}$

$u = -x^2$

$\frac{du}{dx} = -2x \rightarrow dx = \frac{du}{-2x}$




**2.** Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

$2x^2+10 = 4x+16$

$2x^2 - 4x - 6 = 0$

$2(x^2-2x-3) = 0$

$2(x-3)(x+1) = 0$

$x=-1 \quad x=3$

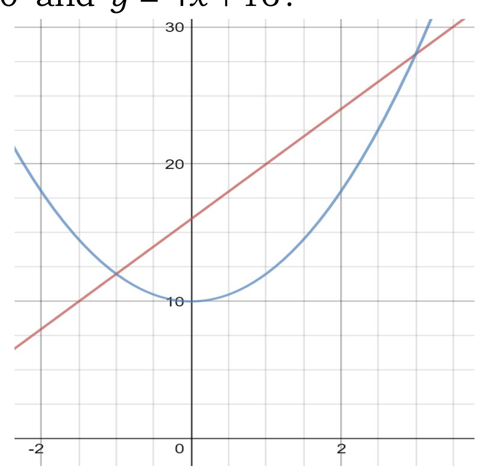
$\int_{-1}^{3} 4x+16-(2x^2+10)\,dx =$

$\int_{-1}^{3} 4x+16-2x^2-10\,dx =$

$\int_{-1}^{3} -2x^2+4x+6\,dx =$

$-\frac{2}{3}x^3 + 2x^2 + 6x\Big]_{-1}^{3} =$

$-\frac{2}{3}(3)^3 + 2(3)^2 + 6(3) - \left(-\frac{2}{3}(-1)^3 + 2(-1)^2 + 6(-1)\right) = \boxed{\frac{64}{3}}$



**3.** Find the area of the region bounded by $y = sin(x)$, $y = cos(x)$, $x = \frac{\pi}{2}$, and $x = 0$.

$\sin x = \cos x$

$x = \frac{\pi}{4}$

$$2\int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx = 2\Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}}$$

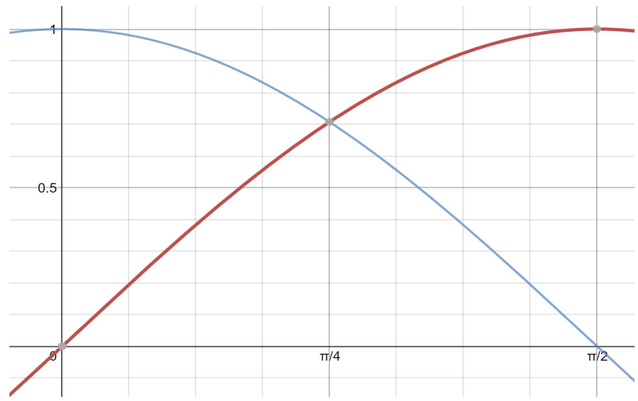
$$= 2\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} - \sin 0 - \cos 0\right)$$

$$= 2\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 0 - 1\right)$$

$$= 2\left(\tfrac{2}{\sqrt{2}} - 1\right)$$

$$= 2\left(\tfrac{2\sqrt{2}}{2} - 1\right)$$

$$\boxed{= 2\sqrt{2} - 2}$$

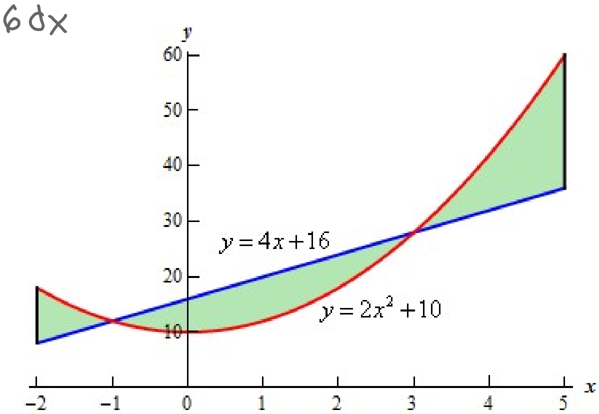



**4.** Set-up, but do not solve, the integral expression that will find the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and $x = 5$.

$$A = \int_{-2}^{1} 2x^2+10-4x-16\,dx + \int_{-2}^{3} 4x+16-2x^2-10\,dx + \int_{3}^{5} 2x^2+10-4x-16\,dx$$

$$\boxed{= \int_{-2}^{1} 2x^2-4x-6\,dx + \int_{-2}^{3} -2x^2+4x+6\,dx + \int_{3}^{5} 2x^2-4x-6\,dx}$$

$$= \frac{160}{3}$$




**5.** Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

$y + 1 = x$

$\frac{1}{2}y^2 - 3 = y + 1$

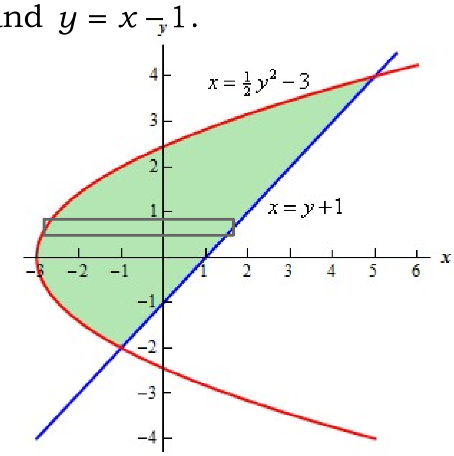
$y^2 - 6 = 2y + 2$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = 4 \quad y = -2$

$\text{Area} = \int_c^d \text{Right} - \text{left}\, dy$




$$\int_{-2}^{4} y + 1 - \tfrac{1}{2}y^2 + 3\,dy = \int_{-2}^{4} -\tfrac{1}{2}y^2 + y + 4\,dy$$

$$= -\tfrac{1}{6}y^3 + \tfrac{1}{2}y^2 + 4y\Big]_{-2}^{4}$$

$$= -\tfrac{1}{6}(4)^3 + \tfrac{1}{2}(4)^2 + 4(4) + \tfrac{1}{6}(-2)^3 - \tfrac{1}{2}(-2)^2 - 4(-2)$$

$$= \frac{-64}{6} + 8 + 16 - \frac{8}{6} - 2 + 8$$

$$= \frac{-32}{3} + 30 - \frac{4}{3} = \frac{-36}{3} + 30 = -12 + 30 = \boxed{18}$$

**6.** Determine the area bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.

$-y^2+10=(y-2)^2$
$-y^2+10=y^2-4y+4$
$0=2y^2-4y-6$
$0=2(y^2-2y-3)$
$0=2(y-3)(y+1)$
$y=-1 \quad y=3$

$\int_{-1}^{3} -y^2+10-(y-2)^2 dy =$

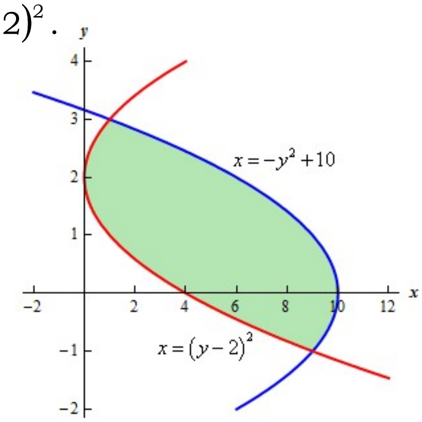
$\int_{-1}^{3} -2y^2+4y+6 dy =$

$-\frac{2}{3}y^3+2y^2+6y\Big]_{-1}^{3} =$

$-\frac{2}{3}(3)^3+2(3)^2+6(3)+\frac{2}{3}(-1)^3-2(-1)^2-6(-1)=$

$-18+18+18-\frac{2}{3}-2+6=$

$\boxed{\frac{64}{3}}$




**7.** Determine the area bounded by $y = 1$ and $y = \cos^2(x)$ on $[0, \pi]$.

$\int_0^{\pi} 1-\cos^2 x dx = \int_0^{\pi} \sin^2 x dx = -\frac{1}{2}\int_0^{\pi} \cos(2x)-1 \, dx$

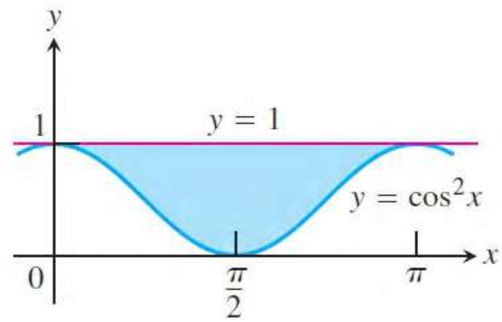
$= -\frac{1}{2}\left[\frac{1}{2}\sin(2x)-x\right]_0^{\pi}$

$= -\frac{1}{2}\left[\frac{1}{2}\sin(2\pi)-\pi-\frac{1}{2}\sin 0+0\right]$

$= -\frac{1}{2}\left[\frac{1}{2}(0)-\pi-\frac{1}{2}(0)\right]$

$\boxed{=\frac{\pi}{2}}$

$\cos(2x)=\cos^2 x-\sin^2 x$
$\cos(2x)=1-2\sin^2 x$
$\frac{\cos(2x)-1}{-2}=\sin^2 x$




**8. Calculator Allowed:** Find the positive value of $k$ such that the area of the region enclosed between the graph of $y = k\cos(x)$ and the graph of $y = kx^2$ is 2.

$k\cos(x)=kx^2$
$k(\cos(x)-x^2)=0$
$x=\pm .824$

$\int_{-.824}^{.824} k\cos x - kx^2 dx = 2$

$k\int_{-.824}^{.824} \cos x - x^2 dx = 2$

$1.0948k=2$

$\boxed{k=1.827}$

**9.** If, for all real numbers $x$, $f(x) = g(x) + 5$, then on any closed interval $[a, b]$, what is the area of the region between the graphs of $f$ and $g$?

$f(x) \geq g(x)$

$\int_a^b f(x)-g(x)dx = \int_a^b f(x)-(f(x)-5)dx$

$=\int_a^b 5dx = 5x\Big]_a^b = \boxed{5b-5a}$

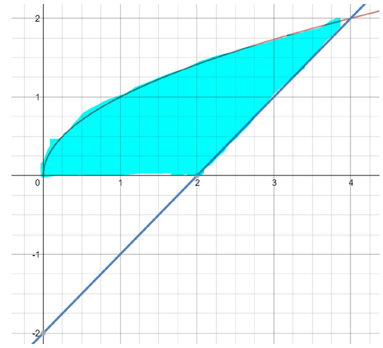
**10**. Find the area of the region $R$ in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$ using a single integral and using the geometry of the region.

$y^2 = x$

$y+2=x$

$$\int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) =$$
$$\frac{2}{3}x^{3/2}\Big]_0^4 - 2 =$$
$$\frac{2}{3}(4)^{3/2} - 2 =$$
$$\frac{2}{3}\cdot 8 - 2 =$$
$$\frac{16}{3} - \frac{6}{3} =$$
$$\boxed{\frac{10}{3}}$$

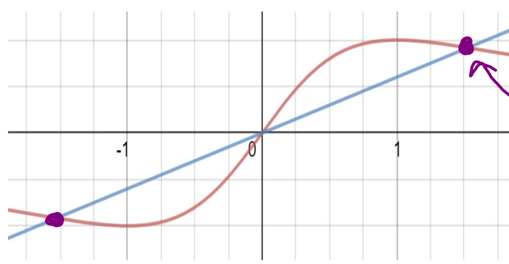
$$\int_0^2 y+2-y^2\,dy =$$
$$\frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big]_0^2 =$$
$$\frac{1}{2}(2)^2+2(2)-\frac{1}{3}(2)^3 =$$
$$2+4-\frac{8}{3} =$$
$$6-\frac{8}{3} =$$
$$\boxed{\frac{10}{3}}$$




**11.** Show that the area of the region enclosed by the curves $y = \dfrac{x}{x^2+1}$ and $y = mx$, where $0 < m < 1$, is equal to $m - ln(m) - 1$.

$$2\int_0^{\sqrt{\frac{1-m}{m}}} \frac{x}{x^2+1} - mx\,dx = 2\left[\frac{1}{2}\ln(x^2+1) - \frac{m}{2}x^2\right]_0^{\sqrt{\frac{1-m}{m}}}$$
$$= \ln\left(\left(\sqrt{\tfrac{1-m}{m}}\right)^2+1\right) - m\left(\sqrt{\tfrac{1-m}{m}}\right)^2 - \ln(1)$$
$$= \ln\left(\frac{1-m}{m}+1\right) - m\left(\frac{1-m}{m}\right)$$
$$= \ln\left(\frac{1-m+m}{m}\right) - 1 + m$$
$$= \ln\left(\frac{1}{m}\right) - 1 + m$$
$$= \ln 1 - \ln m - 1 + m$$
$$= m - \ln(m) - 1$$

$$\frac{x}{x^2+1} = mx$$
$$mx(x^2+1) = x$$
$$mx^3 + mx = x$$
$$mx^3 + mx - x = 0$$
$$x(mx^2 + m - 1) = 0$$
$$x = 0 \qquad mx^2 = 1-m$$
$$x^2 = \frac{1-m}{m}$$
$$x = \pm\sqrt{\frac{1-m}{m}}$$

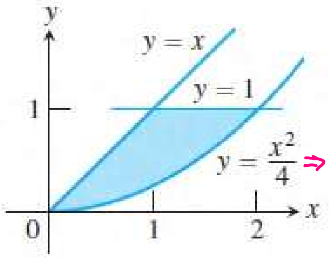



**12.** The area under the cure $y = e^x$ from $x = 0$ to $x = k$ is one. Find the value of $k$.

$$\int_0^k e^x\,dx = 1$$
$$e^x\Big]_0^k = 1$$
$$e^k - e^0 = 1$$
$$e^k - 1 = 1$$
$$e^k = 2$$
$$\boxed{\ln 2 = k}$$

**Directions:** Analytically find the area between the two curves. Calculators not allowed! Show all work that leads to your final answer.

**13.**




$\Rightarrow 4y = x^2$

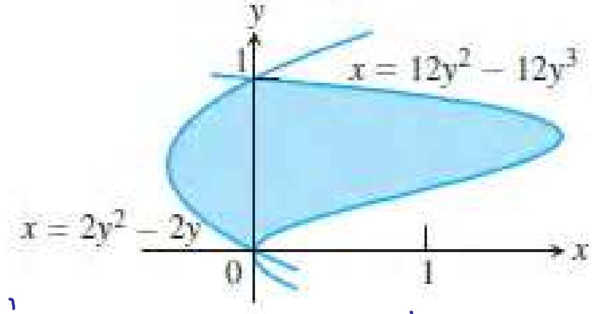
$2\sqrt{y} = x$

$\int_0^1 2\sqrt{y} - y\,dy = \frac{4}{3}y^{3/2} - \frac{1}{2}y^2 \Big]_0^1$

$= \frac{4}{3} - \frac{1}{2}$

$= \frac{8}{6} - \frac{3}{6} = \boxed{\frac{5}{6}}$

**14.**




$\int_0^1 12y^2 - 12y^3 - 2y^2 + 2y\,dy = \int_0^1 10y^2 - 12y^3 + 2y\,dy$

$= \frac{10}{3}y^3 - 3y^4 + y^2 \Big]_0^1 = \frac{10}{3} - 3 + 1 = \frac{10}{3} - 2$

$= \frac{10}{3} - \frac{6}{3} = \boxed{\frac{4}{3}}$

$2y^2 - 2y = 12y^2 - 12y^3$

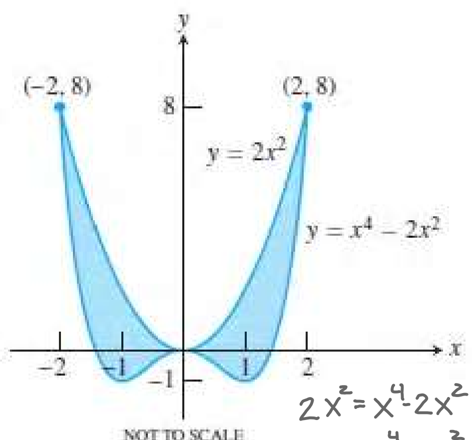
$0 = -12y^3 + 10y^2 + 2y$

$0 = -2y(6y^2 + 5y - 1)$

$0 = -2y(6y+1)(y-1)$

$y = -\frac{1}{6}$ $y = 1$ $y = 0$

**15.**




$2x^2 = x^4 - 2x^2$

$0 = x^4 - 4x^2$

$0 = x^2(x^2 - 4)$

$x = 0$ $x = \pm 2$

$2\int_0^2 2x^2 - x^4 + 2x^2\,dx = 2\int_0^2 4x^2 - x^4\,dx$

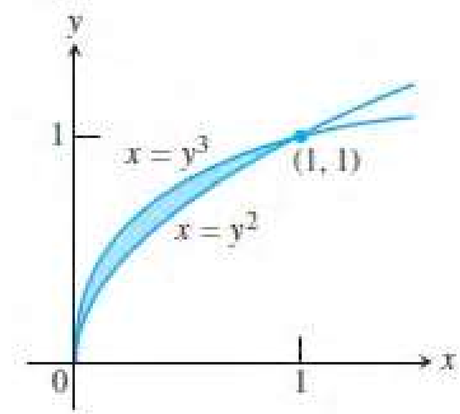
$= 2\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$

$= 2\left[\frac{4}{3}(2)^3 - \frac{1}{5}(2)^5\right]$

$= 2\left(\frac{32}{3} - \frac{32}{5}\right)$

$= 2\left(\frac{160}{15} - \frac{96}{15}\right) = \boxed{\frac{128}{15}}$

**16.**



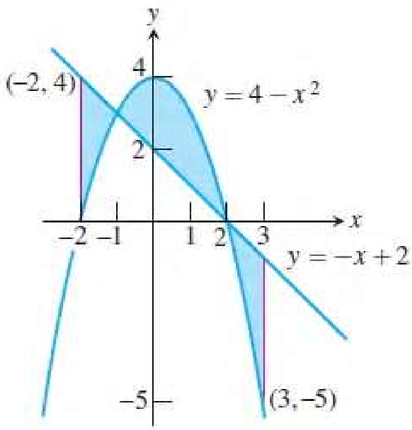

$\int_0^1 y^2 - y^3\,dy = \left[\frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1$

$= \frac{1}{3} - \frac{1}{4}$

$= \frac{4}{12} - \frac{3}{12}$

$\boxed{= \frac{1}{12}}$

**17.**




$4-x^2=-x+2$

$0=x^2-x-2$

$0=(x-2)(x+1)$

$x=-1,+2$

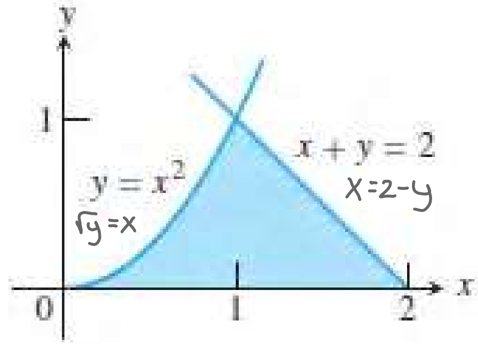
$$\int_{-2}^{-1} -x+2-4+x^2\,dx+\int_{-1}^{2} 4-x^2+x-2\,dx+\int_{2}^{3} -x+2-4+x^2\,dx=$$

$$\int_{-2}^{-1} x^2-x-2\,dx+\int_{-1}^{2} -x^2+x+2\,dx+\int_{2}^{3} x^2-x-2\,dx=$$

$$\left[\tfrac{1}{3}x^3-\tfrac{1}{2}x^2-2x\right]_{-2}^{-1}+\left[-\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+2x\right]_{-1}^{2}+\left[\tfrac{1}{3}x^3-\tfrac{1}{2}x^2-2x\right]_{2}^{3}=$$

$$\boxed{\frac{49}{6}}$$

**18.**




$2-y=\sqrt{y}$

$4-4y+y^2=y$

$y^2-5y+4=0$

$(y-1)(y-4)=0$

$y=1,\ \cancel{y=4}$

$$\int_0^1 2-y-\sqrt{y}\,dy=$$

$$\left[2y-\tfrac{1}{2}y^2-\tfrac{2}{3}y^{3/2}\right]_0^1=$$

$$2-\frac{1}{2}-\frac{2}{3}=\frac{12}{6}-\frac{3}{6}-\frac{4}{6}\boxed{=\frac{5}{6}}$$